College entrance exam and science high school entrance test tips. Conquer UPCAT, ACET, USTET, DLSUCET, PSHS-NCE, and other entrance tests.

GEOMETRY POINTERS

Euclidean geometry axioms

As we have noted above, there is a set of the axioms – properties, that are considered in geometry as main ones and are adopted without a proof . Now, after introducing some initial notions and definitions we can consider the following sufficient set of the axioms, usually used in plane geometry.

Axiom of belonging. *Through any two points in a plane it is possible to draw a straight line, and besides only one.*

Axiom of ordering. *Among any three points placed in a straight line, there is no more than one point placed between the two others.*

Axiom of congruence (equality) of segments and angles. *If two segments (angles) are congruent to the third one, then they are congruent to each other.*

Axiom of parallel straight lines. *Through any point placed outside of a straight line it is possible to draw another straight line, parallel to the given line, and besides only one.*

Axiom of continuity (Archimedean axiom). *Let AB and CD be two some segments; then there is a finite set of such points* A_1 , A_2 , … , A_n , *placed in the straight line AB, that segments* AA_1 , A_1A_2 , … , $A_{n-1}A_n$ *are congruent to segment CD, and point B is placed between A and* A_n .

We emphasize, that replacing one of these axioms by another, turns this axiom into a theorem, requiring a proof. So, instead of the axiom of parallel straight lines we can use as an axiom the property of triangle angles ("the sum of triangle angles is equal to 180 deg"). But then we should to prove the property of parallel lines.

Straight line

A general equation of straight line:

$$Ax + By + C = 0 ,$$

where A and B aren't equal to zero simultaneously.

Coefficients A and B are coordinates of ***normal vector*** of the straight line (i.e. vector, perpendicular to the straight line). At $A = 0$ straight line is parallel to the axis OX, at $B = 0$ straight line is parallel to the axis OY.

At $B \neq 0$ we receive ***an equation of straight line with a slope***:

$$y = -\frac{A}{B}x - \frac{C}{B} = mx + k.$$

An equation of the straight line, going through the point (x_0, y_0) and not parallel to the axis OY:

$$y - y_0 = m(x - x_0),$$

where m is ***a slope***, equal to tangent of an angle between the straight line and the positive direction of the axis OX.

At $A \neq 0$, $B \neq 0$ and $C \neq 0$ we receive ***an equation of straight line in segments on axes***:

$$\frac{x}{a} + \frac{y}{b} = 1,$$

where $a = -C/A$, $b = -C/B$. This line goes through the points (a, 0) and (0, b), i.e. it cuts off segments a and b long on the coordinate axes.

An equation of straight line going through two different points (x_1, y_1) and (x_2, y_2):

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}, \quad \text{at } x_1 \neq x_2,\ y_1 \neq y_2.$$

Parallel straight lines

Two straight lines AB and CD (Fig.11) are called *parallel straight lines*, if they lie in the same plane and don't intersect however long they may be continued. The designation: AB∥ CD. All points of one line are equidistant from another line. All straight lines, parallel to one straight line are parallel between themselves. It's adopted that an angle between parallel straight lines is equal to zero. An angle between two parallel *rays* is equal to zero, if their directions are the same and 180 deg, if the directions are opposite. All perpendiculars (AB, CD, EF, and Fig.12) to the one straight line KM are parallel between themselves. Inversely, the straight line KM, which is perpendicular to one of parallel straight lines, is perpendicular to all others. A length of perpendicular segment, concluded between two parallel straight lines, is a *distance* between them.

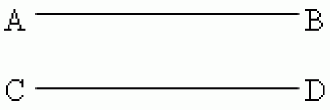

Fig. 11

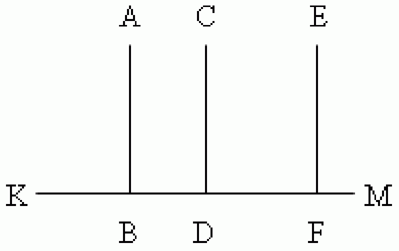

Fig. 12

At intersecting two parallel straight lines by the third line, eight angles are formed (Fig.13), which are called two-by-two:

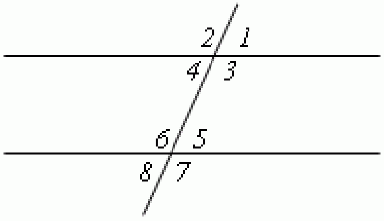

Fig. 13

1) *corresponding angles* (*1* and *5; 2* and *6; 3* and *7; 4* and *8*); these angles are equal two-by-two: ($\angle 1 = \angle 5$; $\angle 2 = \angle 6$; $\angle 3 = \angle 7$; $\angle 4 = \angle 8$);
2) *alternate interior angles* (*4* and *5; 3* and *6*); they are equal two-by-two;
3) *alternate exterior angles* (*1* and *8; 2* and *7*); they are equal two-by-two;
4) *one-sided interior angles* (*3* and *5; 4* and *6*); a sum of them two-by-two is equal to180 deg ($\angle 3 + \angle 5$ = 180 deg; $\angle 4 + \angle 6$ = 180 deg);
5) *one-sided exterior angles* (*1* and *7; 2* and *8*); a sum of them two-by-two is equal to180 deg ($\angle 1 + \angle 7$ = 180 deg; $\angle 2 + \angle 8$ = 180 deg).

Angles with *correspondingly parallel sides* either are equal one to another, (if both of them are acute or both are obtuse, $\angle 1 = \angle 2$, Fig.14), or sum of them is 180 deg ($\angle 3 + \angle 4$ = 180 deg, Fig.15).

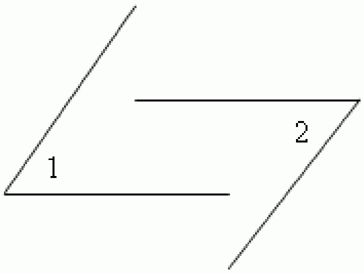

Fig. 14

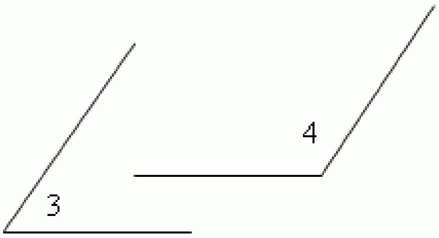

Fig. 15

Angles with *correspondingly perpendicular sides* are also either equal one to another (if both of them are acute or both are obtuse), or sum of them is 180 deg.

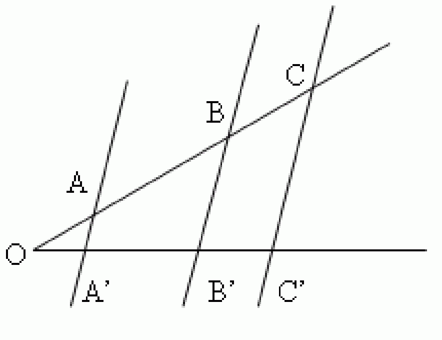

Fig. 16

Thales' theorem. *At intersecting sides of an angle by parallel lines* (Fig.16), *the angle sides are divided into the proportional segments:*

$$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'} .$$

Angles

Angle is a geometric figure (Fig.1), formed by two rays OA and OB (*sides of an angle*), going out of the same point O (a *vertex of an angle*).

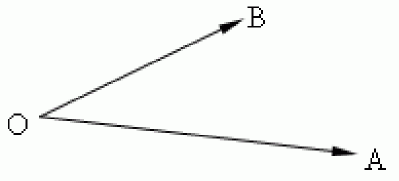

Fig. 1

An angle is signed by the symbol ∠and three letters, marking ends of rays and a vertex of an angle: ∠ AOB (moreover, a vertex letter is placed in the middle). A *measure of an angle* is a value of a turn around a vertex O, that transfers a ray OA to the position OB. Two units of angles measures are widely used: a *radian* and a *degree*. About a radian measure see below in the point "A length of arc" and also in the section "Trigonometry".
A *degree measure*. Here a unit of measurement is a *degree* (its designation is ° or *deg*) – a turn of a ray by the 1/360 part of the one complete revolution. So, the complete revolution of a ray is equal to 360 deg. One degree is divided by 60 *minutes* (a designation is ' or *min*); one minute – correspondingly by 60 *seconds* (a designation is " or *sec*). An angle of 90 deg (Fig.2) is called a *right* or *direct* angle; an angle lesser than 90 deg (Fig.3), is called an *acute* angle; an angle greater than 90 deg (Fig.4), is called an *obtuse* angle.

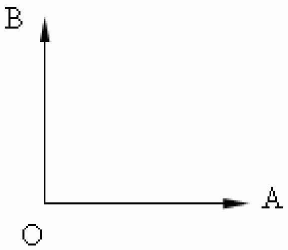

Fig. 2

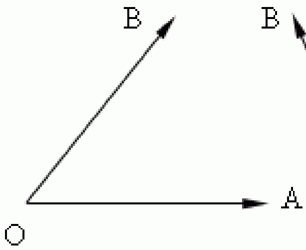

Fig. 3

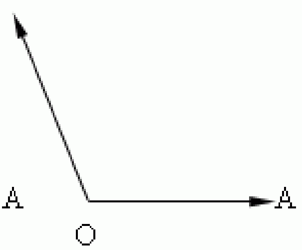

Fig. 4

Straight lines, forming a right angle, are called *mutually perpendicular* lines. If the straight lines AB and MK are perpendicular, this is signed as: AB ⊥MK.

Signs of angles. An angle is considered as *positive*, if a rotation is executed *opposite a clockwise*, and *negative* – otherwise. For example, if the ray OA displaces to the ray OB as shown on Fig.2, then ∠AOB = + 90 deg; but on Fig.5 ∠AOB = – 90 deg.

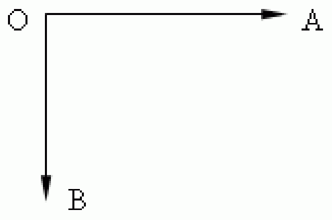

Fig. 5

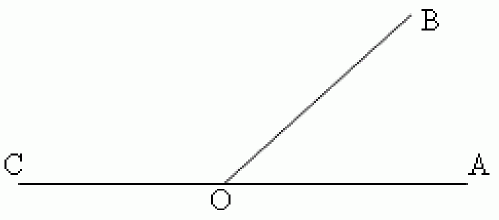

Fig. 6

Supplementary (adjacent) angles (Fig.6) – angles AOB and COB, having the common vertex O and the common side OB; other two sides OA and OC form a continuation one to another. So, a sum of supplementary (adjacent) angles is equal to 180 deg.
Vertically opposite (vertical) angles (Fig.7) – such two angles with a common vertex, that sides of one angle are continuations of the other: ∠AOB and ∠ COD (and also ∠AOC and ∠DOB) are vertical angles.

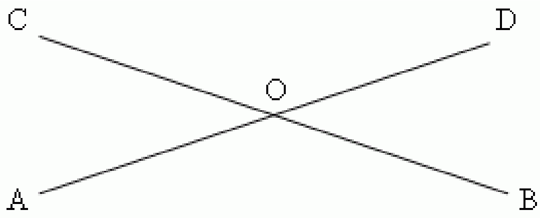

Fig. 7

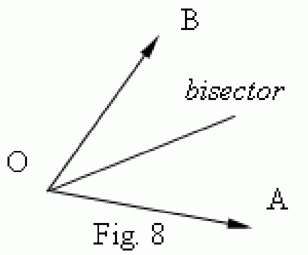

Fig. 8

A *bisector* of an angle is a ray, dividing the angle in two (Fig.8). Bisectors of vertical angles (OM and ON, Fig.9) are continuations one of the other. Bisectors of supplementary angles (OM and ON, Fig.10) are mutually perpendicular lines.

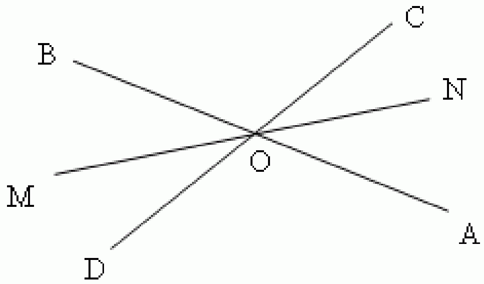

Fig. 9

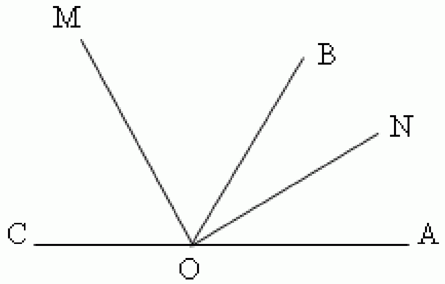

Fig. 10

The property of an angle bisector: *any point of an angle bisector is placed by the same distance from the angle sides*.

Triangle

Triangle is a polygon with three sides (or three angles). Sides of triangle are signed often by small letters, corresponding to designations of opposite vertices, signed by capital letters.

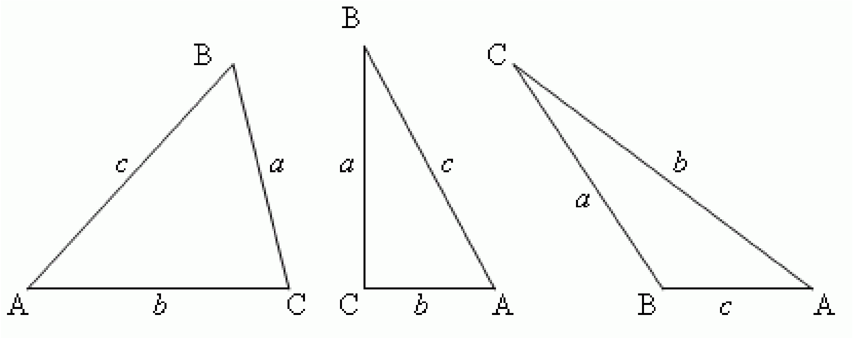

Fig. 20 Fig. 21 Fig. 22

If all the three angles are acute (Fig.20), then this triangle is an *acute-angled triangle*; if one of the angles is right (∠C, Fig.21), then this triangle is a *right-angled triangle*; sides *a, b*, forming a right angle, are called *legs*; side *c*, opposite to a right angle, called a *hypotenuse*; if one of the angles is obtuse (∠B, Fig.22), then this triangle is an *obtuse-angled triangle.*

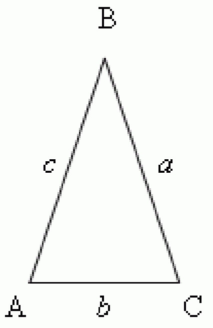

Fig. 23

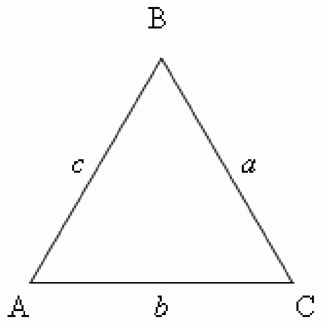

Fig. 24

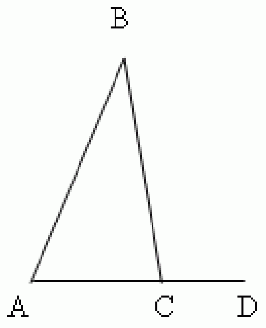

Fig. 25

A triangle ABC is an *isosceles triangle* (Fig.23), if the *two* of its sides are equal ($a = c$); these equal sides are called *lateral* sides, the third side is called a *base* of triangle. A triangle ABC is an *equilateral triangle* (Fig.24), if all of its sides are equal
($a = b = c$). In general case ($a \neq b \neq c$) we have a *scalene triangle.*

Main properties of triangles. In *any triangle:*

1. *An angle, lying opposite the greatest side, is also the greatest angle, and inversely.*

2. *Angles, lying opposite the equal sides, are also equal, and inversely.* In particular, all angles in an *equilateral triangle* are also equal.

3. *A sum of triangle angles is equal to 180 deg.*

   From the two last properties it follows, that each angle in an equilateral triangle is equal to 60 deg.

4. Continuing one of the triangle sides (AC , Fig. 25), we receive an *exterior* angle $\angle BCD$. *An exterior angle of a triangle is equal to a sum of interior angles, not supplementary with it*: $\angle BCD = \angle A + \angle B$.

5. *Any side of a triangle is less than a sum of two other sides and more than their difference* ($a < b + c$, $a > b - c$; $b < a + c$, $b > a - c$; $c < a + b$, $c > a - b$).

Theorems about congruence of triangles.

Two triangles are congruent, if they have accordingly equal:

a) *two sides and an angle between them;*

b) *two angles and a side, adjacent to them;*

c) *three sides.*

Theorems about congruence of right-angled triangles.

Two *right-angled* triangles are congruent, if one of the following conditions is valid:

1) their legs are equal;

2) a leg and a hypotenuse of one of triangles are equal to a leg and a hypotenuse of another;

3) a hypotenuse and an acute angle of one of triangles are equal to a hypotenuse and an acute angle of another;

4) a leg and an adjacent acute angle of one of triangles are equal to a leg and an adjacent acute angle of another;

5) a leg and an opposite acute angle of one of triangles are equal to a leg and an opposite acute angle of another.

Remarkable lines and points of triangle.

Altitude (height) of a triangle is a *perpendicular, dropped from any vertex to an opposite side* (or to its continuation). This side is called a *base* of triangle in this case. Three heights of triangle always intersect in one point, called an orthocenter of a triangle. An orthocenter of an acute-angled triangle (point O, Fig.26) is placed inside of the triangle; and an orthocenter of an obtuse-angled triangle (point O, Fig.27) – outside of the triangle; an orthocenter of a right-angled triangle coincides with a vertex of the right angle.

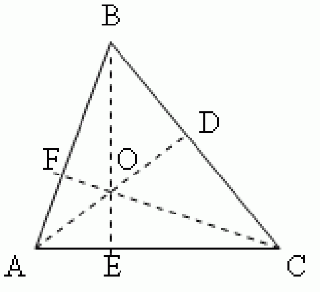

Fig. 26

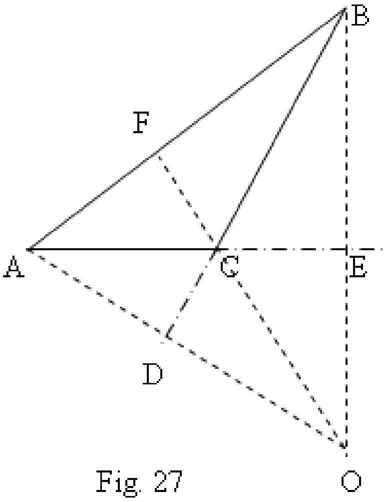

Fig. 27

Median is a *segment, joining any vertex of triangle and a midpoint of the opposite side. Three medians of triangle* (AD, BE, CF, Fig.28) *intersect in one point* O (always lied inside of a triangle), which is a *center of gravity* of this triangle. This point divides each median by ratio 2:1, considering from a vertex.

Bisector is a segment of the angle bisector, from a vertex to a point of intersection with an opposite side. *Three bisectors of a triangle* (AD, BE, CF, Fig.29) *intersect in the one point* (always lied inside of triangle), *which is a center of an inscribed circle* (see the section "Inscribed and circumscribed polygons").

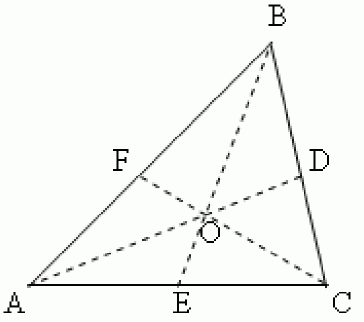

Fig. 28

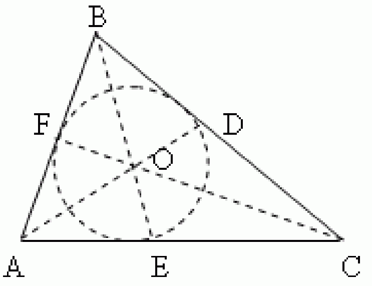

Fig. 29

A bisector divides an opposite side into two parts, proportional to the adjacent sides; for instance, on Fig.29 AE : CE = AB : BC .

Midperpendicular is a perpendicular, drawn from a middle point of a segment (side).Three midperpendiculars of a triangle (ABC, Fig.30), each drawn through the middle of its side (points K, M, N, Fig.30), intersect in one point O, which is *a center of circle, circumscribed* around the triangle (circumcircle).

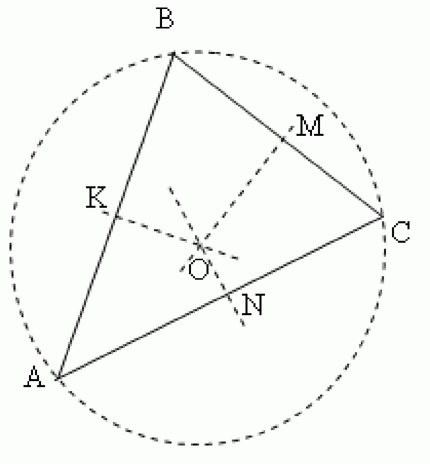

Fig. 30

In an acute-angled triangle this point lies inside of the triangle; in an obtuse-angled triangle - outside of the triangle; in a right-angled triangle - in the middle of the hypotenuse. An orthocenter, a center of gravity, a center of an inscribed circle and a center of a circumcircle coincide only in an equilateral triangle.

Pythagorean theorem. *In a right-angled triangle a square of the hypotenuse length is equal to a sum of squares of legs lengths.*

A proof of Pythagorean theorem is clear from Fig.31. Consider a right-angled triangle ABC with legs *a, b* and a hypotenuse *c*.

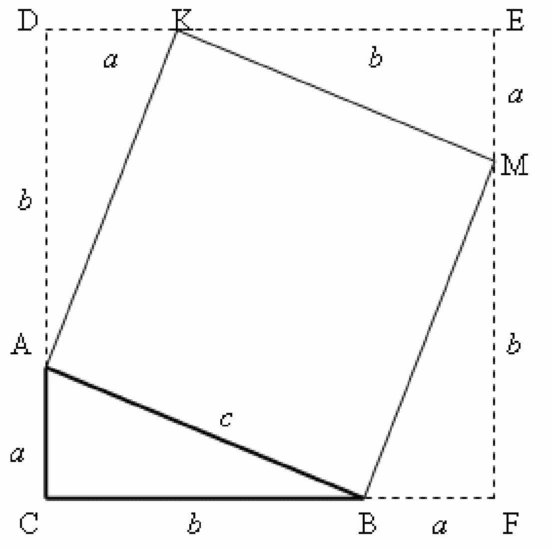

Fig. 31

Build the square AKMB, using hypotenuse AB as its side. Then continue sides of the right-angled triangle ABC so, to receive the square CDEF, the side length of which is equal to $a + b$. Now it is clear, that an area of the square CDEF is equal to $(a + b)^2$. On the other hand, this area is equal to a sum of areas of four right-angled triangles and a square AKMB, that is

$$c^2 + 4 (ab / 2) = c^2 + 2 ab ,$$

hence,

$$c^2 + 2 ab = (a + b)^2,$$

and finally, we have:

$$c^2 = a^2 + b^2.$$

Relation of sides' lengths for arbitrary triangle.

In general case (for any triangle) we have:

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C,$$

where C – an angle between sides a and b.

Parallelogram and trapezoid

Parallelogram (ABCD, Fig.32) is a quadrangle, opposite sides of which are two-by-two parallel.

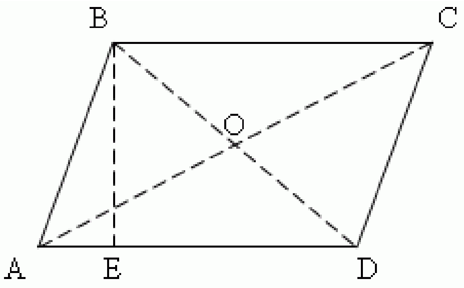

Fig. 32

Any two opposite sides of a parallelogram are called *bases*, a distance between them is called a *height* (BE, Fig.32).

Properties of a parallelogram.

1. *Opposite sides of a parallelogram are equal* (AB = CD, AD = BC).

2. *Opposite angles of a parallelogram are equal* ($\angle A = \angle C$, $\angle B = \angle D$).

3. *Diagonals of a parallelogram are divided in their intersection point into two* (AO = OC, BO = OD).

4. *A sum of squares of diagonals is equal to a sum of squares of four sides*: $AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + AD^2$.

Signs of a parallelogram.

A quadrangle is a parallelogram, if one of the following conditions takes place:

1. *Opposite sides are equal two-by-two* (AB = CD, AD = BC).

2. *Opposite angles are equal two-by-two* (∠A = ∠C, ∠B = ∠D).

3. *Two opposite sides are equal and parallel* (AB = CD, AB || CD).

4. *Diagonals are divided in their intersection point into two* (AO = OC, BO = OD).

Rectangle.

If one of angles of parallelogram is right, then all angles are right (why ?). This parallelogram is called a *rectangle* (Fig.33).

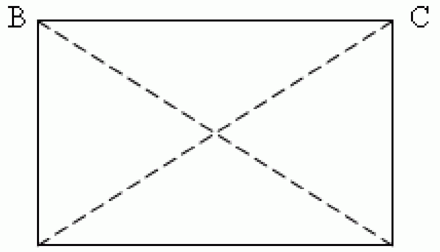

Fig. 33

<u>*Main properties of a rectangle.*</u>

Sides of rectangle are its heights simultaneously.

Diagonals of a rectangle are equal: AC = BD.

A square of a diagonal length is equal to a sum of squares of its sides' lengths (see above Pythagorean theorem):

$$AC^2 = AD^2 + DC^2.$$

Rhombus. If all sides of parallelogram are equal, then this parallelogram is called a *rhombus* (Fig.34) .

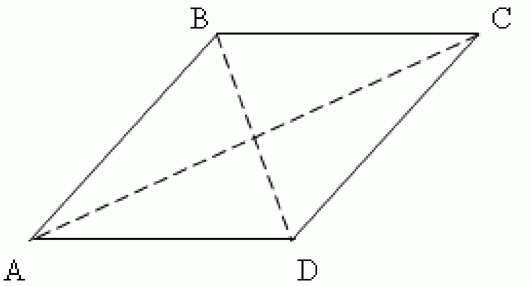

Fig. 34

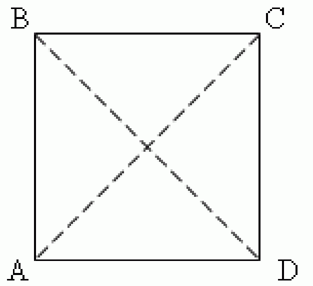

Fig. 35

Diagonals of a rhombus are mutually perpendicular ($AC \perp BD$) *and divide its angles into two* ($\angle DCA = \angle BCA$, $\angle ABD = \angle CBD$ etc.).

Square is a parallelogram with right angles and equal sides (Fig.35). *A square is a particular case of a rectangle and a rhombus simultaneously; so, it has all their above mentioned properties.*

Trapezoid is a quadrangle, two opposite sides of which are parallel (Fig.36).

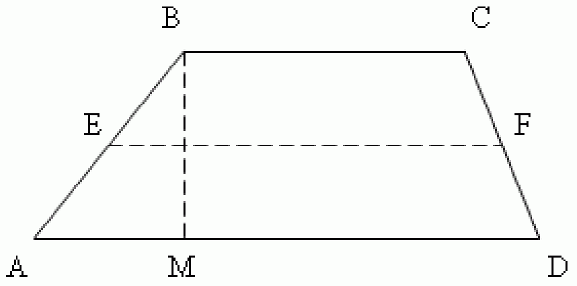

Fig. 36

Here $AD \parallel BC$. Parallel sides are called *bases* of a trapezoid, the two others (AB and CD) – *lateral sides.* A distance between bases (BM) is a *height.* The segment EF, joining midpoints E and F of the lateral sides, is called a *midline* of a trapezoid.
A midline of a trapezoid is equal to a half-sum of bases:

$$EF = \frac{AD + BC}{2}$$

and parallel to them: EF ‖ AD and EF ‖ BC.
A trapezoid with equal lateral sides (AB = CD) is called an *isosceles*trapezoid. *In an isosceles trapezoid angles by each base, are equal* (∠A = ∠D, ∠B = ∠C). A parallelogram can be considered as a particular case of trapezoid.

Midline of a triangle is a segment, *joining midpoints of lateral sides* of a triangle. *A midline of a triangle is equal to half of its base and parallel to it.*This property follows from the previous part, as triangle can be considered as a limit case ("degeneration") of a trapezoid, when one of its bases transforms to a point.

Volumes and areas of body surfaces

Designations: *V – a volume; S – a base area; S_{lat} – a lateral surface area; P – a full surface area; h – a height; a, b, c – dimensions of a right angled parallelepiped; A – an apothem of a regular pyramid and a regular truncated pyramid; L – a generatrix of a cone; p – a perimeter or a circumference of a base; r – a radius of a base; d – a diameter of a base; R – a radius of a ball; D – a diameter of a ball; indices 1 and 2 are related to radii, diameters, perimeters and areas of upper and lower bases of truncated prism and pyramid.*

A prism (right and oblique) *and a parallelepiped:*

$$V = Sh\ .$$

A right prism:

$$S_{lat} = ph\ .$$

A right angled parallelepiped:

$$V = abc\ ; \quad P = 2\ (\ ab + bc + ab\ )\ .$$

A cube:

$$V = a^3\ ;\quad P = 6a^2\ .$$

A pyramid (regular and irregular) *:*

$$V = \frac{1}{3} Sh\ .$$

A regular pyramid:

$$S_{lat} = \frac{1}{2}\, pA\ .$$

A truncated pyramid (regular and irregular) *:*

$$V = \frac{1}{3}\left(S_1 + \sqrt{S_1 S_2} + S_2 \right) h$$

A regular truncated pyramid:

$$S_{lat} = \frac{1}{2}\left(p_1 + p_2 \right) h\ .$$

A circular cylinder (right and oblique)*:*

$$V = Sh = \pi r^{2} h = \frac{1}{4} \pi d^{2} h .$$

A round cylinder :

$$S_{lat} = 2\pi rh = \pi dh .$$

A circular cone (round and oblique)*:*

$$V = \frac{1}{3} Sh = \frac{1}{3} \pi r^{2} h = \frac{1}{12} \pi d^{2} h .$$

A round cone:

$$S_{lat} = \frac{1}{2} pL = \pi rL = \frac{1}{2} \pi dL .$$

A truncated circular cone (round and oblique)*:*

$$V = \frac{1}{3} \pi h (r_1^2 + r_1 r_2 + r_2^2) = \frac{1}{12} \pi h (d_1^2 + d_1 d_2 + d_2^2) .$$

A truncated round cone:

$$S_{lat} = \pi\,(r_1 + r_2)\,L = \frac{1}{2}\pi\,(d_1 + d_2)\,L\,.$$

A sphere (ball):

$$V = \frac{4}{3}\pi R^3 = \frac{1}{6}\pi\, D^3\;; \quad P = 4\pi R^2 = \pi D^2\;.$$

A hemisphere:

$$V = \frac{2}{3}\pi R^3 = \frac{1}{12}\pi\, D^3\;; \quad S = \pi R^2 = \frac{1}{4}\pi D^2\;;$$

$$S_{lat} = 2\pi R^2 = \frac{1}{2}\pi D^2\,; \quad P = 3\pi R^2 = \frac{3}{4}\pi D^2\;.$$

A spherical segment:

$$V = \pi h^2\,(R - h/3\,) = \frac{\pi h}{6}(h^2 + 3r^2)\,;$$

$$S_{lat} = 2\pi Rh = \pi\,(r^2 + h^2);\; P = \pi\,(2r^2 + h^2)\,.$$

A spherical layer:

$$V=\frac{1}{6}\pi h^3+\frac{1}{2}\pi\left(r_1^2+r_2^2\right)h\ ;\quad S_{lat}=2\pi Rh\,.$$

A spherical sector:

$$V=\frac{2}{3}\pi R\bar{h}\,,$$

here h – a height of a segment, contained in the sector.

A hollow ball:

$$V=\frac{4}{3}\pi\left(R_1^3-R_2^3\right)=\frac{1}{6}\pi\left(D_1^3-D_2^3\right);\quad P=4\pi\left(R_1^2+R_2^2\right)=\pi\left(D_1^2+D_2^2\right).$$

here R_1 , R_2 , D_1 , D_2 – radii and diameters of external and internal spherical surfaces correspondingly.

Circle

A circle (Fig.1) is a locus of points, equidistant from the given point O, called a ***center of circle***, at the distance R. A number $R > 0$ is called ***a radius of circle***.

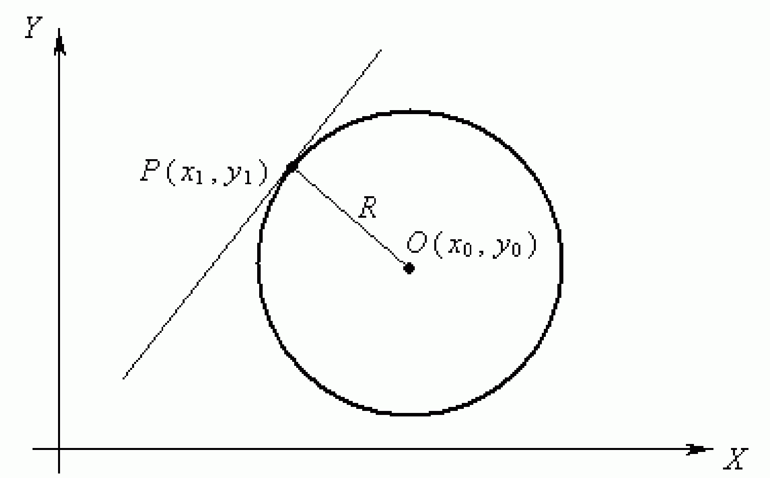

Fig. 1

An equation of circle of radius R with a center in a point $O(x_0, y_0)$ is:

$$(x - x_0)^2 + (y - y_0)^2 = R^2 .$$

If *a center of the circle coincides with the origin of coordinates*, then an equation of circle becomes:

$$x^2 + y^2 = R^2 .$$

Let $P(x_1, y_1)$ be a point of the circle (Fig.1), then ***an equation of tangent line to circle*** in the given point is:

$$(x_1 - x_0)(x - x_0) + (y_1 - y_0)(y - y_0) = R^2 .$$

A tangency condition of a straight line $y = mx + k$ ***and a circle*** $x^2 + y^2 = R^2$:

$$k^2 / (1 + m^2) = R^2 .$$