INTERMEDIATE ALGEBRA POINTERS

Powers and roots

Operations with powers.

1. At multiplying of powers with the same base their exponents are added:

$$a^m \cdot a^n = a^{m+n}.$$

2. At dividing of powers with the same base their exponents are subtracted:

$$\frac{a^m}{a^n} = a^{m-n}.$$

3. A power of product of two or some factors is equal to a product of powers of these factors:

$$(abc\ldots)^n = a^n \cdot b^n \cdot c^n \ldots$$

4. A power of a quotient (fraction) is equal to a quotient of powers of a dividend (numerator) and a divisor (denominator):

$$(a/b)^n = a^n / b^n.$$

5. At raising of a power to a power their exponents are multiplied:

$$(a^m)^n = a^{mn}.$$

All above mentioned formulas are read and executed in both directions – from the left to the right and back.
E x a m p l e . $(2 \cdot 3 \cdot 5 / 15)^2 = 2^2 \cdot 3^2 \cdot 5^2 / 15^2 = 900 / 225 = 4$.

Operations with roots. In all below mentioned formulas a symbol $\sqrt{}$ means an *arithmetical root* (all radicands are considered here only positive).

1. A root of product of some factors is equal to a product of roots of these factors:

$$\sqrt[n]{abc\ldots} = \sqrt[n]{a} \cdot \sqrt[n]{b} \cdot \sqrt[n]{c} \ldots.$$

2. A root of a quotient is equal to a quotient of roots of a dividend and a divisor:

$$\sqrt[n]{a/b} = \sqrt[n]{a} / \sqrt[n]{b}$$

3. At raising a root to a power it is sufficient to raise a radicand to this power:

$$\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

4. If to increase a degree of a root by n times and to raise simultaneously its radicand to the n-th power, the root value doesn't change:

$$\sqrt[n]{a} = \sqrt[mn]{a^m}.$$

5. If to decrease a degree of a root by n times and to extract simultaneously the n-th degree root of the radicand, the root value doesn't change:

$$\sqrt[n]{a} = \sqrt[n:m]{\sqrt[m]{a}}.$$

$$\text{Example. } \sqrt[6]{64} = \sqrt[6]{8 \cdot 8} = \sqrt[6]{8} \cdot \sqrt[6]{8} = \sqrt[3 \cdot 2]{8^2} = \sqrt[3]{8} = 2.$$

Widening of the power notion. Till now we considered only natural exponents of powers; but operations with powers and roots can result also to *negative*, *zero* and *fractional exponents*. All these exponents of powers require to be defined.

Negative exponent of a power. *A power of some number with a negative (integer) exponent is defined as unit divided by the power of the same number with the exponent equal to an absolute value of the negative exponent:*

$$a^{-n} = \frac{1}{a^n}.$$

Now the formula $a^m : a^n = a^{m-n}$ may be used not only if m is more than n, but also for a case if m is less than n.

E x a m p l e . $a^4 : a^7 = a^{4-7} = a^{-3}$.

If we want the formula $a^m/a^n = a^{m-n}$ to be valid at $m = n$ we need the definition of zero exponent of a power.

Zero exponent of a power. *A power of any non-zero number with zero exponent is equal to 1.*

E x a m p l e s: $2^0 = 1,\ (-5)^0 = 1,\ (-3/5)^0 = 1.$

Fractional exponent of a power. *To raise a real number a to a power with an exponent m / n it is necessary to extract the n-th degree root from the m-th power of this number a:*

$$a^{m/n} = \sqrt[n]{a^m}$$

E x a m p l e : $4^{3/2} = \sqrt{4^3} = \sqrt{64} = 8$

About meaningless expressions. There are some expressions:

Case 1. $\frac{a}{0}$, $\frac{a}{0} = x$,

where $a \neq 0$, *doesn't exist.*

Really, if to assume that where x – some number, then according to the definition of a division we have: $a = 0 \cdot x$, i.e. $a = 0$, but this result contradicts to the condition: $a \neq 0$.

Case 2. $\frac{0}{0}$ is *any number.*

Really, if to assume that this expression is equal to some number x, then according to the definition of a division: $0 = 0 \cdot x$. But this equality is valid at *any number* x, which was to be proved.

Case 3.

If to assume, that rules of operations with powers are spread to powers with a *zero* base, then

0^0 is *any number*.

Really,

$$0^0 = 0^{3-3} = \frac{0^3}{0^3} = \frac{0}{0} = \textit{any number}.$$

E x a m p l e . Solve the equation: $\dfrac{|x|}{x} = 1$.

S o l u t i o n . Consider the three main cases:

1) $x = 0$ – this value doesn't satisfy the equation (Why ?) ;

2) at $x > 0$ we receive: $x / x = 1$, i.e. $1 = 1$, hence, x – any number, but taking into consideration that in this case $x > 0$, the answer is: $x > 0$;

3) at $x < 0$ we receive: $-x / x = 1$, i.e. $-1 = 1$, and the answer is: there is no solution in this case. So, the answer: $x > 0$.

Formulas of abridged multiplication

From the rules of multiplication of sums and polynomials the following seven formulas of abridged multiplication can be easily received. *It is necessary to know them by heart,* as they are used in most of problems in mathematics.

[1] $(a + b)^2 = a^2 + 2ab + b^2$,

[2] $(a - b)^2 = a^2 - 2ab + b^2$,

[3] $(a + b)(a - b) = a^2 - b^2$,

[4] $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$,

[5] $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$,

[6] $(a + b)(a^2 - ab + b^2) = a^3 + b^3$,

[7] $(a-b)(a^2+ab+b^2)=a^3-b^3.$

Example:

Factoring of polynomials

In general case factoring of a polynomial is not always possible. But there are some cases, when it can be executed.

1. If all terms of a polynomial contain as a factor the same expression, it is possible to take it out of brackets (see above).

2. Sometimes grouping terms of a polynomial into brackets, one can find a common expression inside the brackets, the expression may be taken out of the brackets as a common factor, and after this the same expression will be inside all brackets Then this expression must also be taken out of the brackets and the polynomial will be factored.

   Example: $ax+bx+ay+by=(ax+bx)+(ay+by)=$

   $$=x(a+b)+y(a+b)=(x+y)(a+b).$$

3. Sometimes including of new, mutually cancelled terms, helps to factor a polynomial.

   Example: $y^2-b^2=y^2+yb-yb-b^2=(y^2+yb)-(yb+b^2)=$

   $$=y(y+b)-b(y+b)=(y+b)(y-b).$$

4. Usage of the formulas of abridged multiplication.

Factoring of a quadratic trinomial

Each quadratic trinomial ax^2+bx+c *can* be resolved to factors of the first degree by the next way. Solve the quadratic equation

$$ax^2+bx+c=0$$

If x_1 and x_2 are the roots of this equation, then

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

This affirmation can be proved using either formula for roots of a non-reduced quadratic equation. (Check it, please!) .

E x a m p l e. Resolve to the first degree factors the trinomial: $2x^2 - 4x - 6$.

S o l u t i o n . At first we solve the equation: $2x^2 - 4x - 6 = 0$. Its roots are: $x_1 = -1$ and $x_2 = 3$. Hence, $2x^2 - 4x - 6 = 2(x + 1)(x - 3)$. (Open the brackets and check the result, please).

Equations of higher degrees

1. Some kinds of the higher degrees equations may be solved using a quadratic equation. Sometimes one can resolve the left-hand side of equation to factors, each of them is a polynomial of the degree not higher than second. Then, equaling each of them to zero and solving all these quadratic and / or linear equations, we'll receive all roots of the original equation.

   E x a m p l e . Solve an equation: $3x^4 + 6x^3 - 9x^2 = 0$.

   S o l u ti o n . Resolve the left-hand side of this equation to factors:

   $$x^2(3x^2 + 6x - 9) .$$

   Solve the equation: $x^2 = 0$; it has two equal roots: $x_1 = x_2 = 0$.
   Now we solve the equation: $3x^2 + 6x - 9 = 0$, and receive:
   $x_3 = 1$ and $x_4 = -3$. Thus, the original equation has four roots:
   $x_1 = x_2 = 0$; $x_3 = 1$; $x_4 = -3$.
2. If an equation has the shape:

$$ax^{2n} + bx^n + c = 0 ,$$

it is reduced to an quadratic equation by the exchange:

$$x^n = z ;$$

really, after this exchange we receive: $az^2 + bz + c = 0$.

E x a m p l e . Consider the equation:

$$x^4 - 13x^2 + 36 = 0 .$$

Exchange: $x^2 = z$. After this we receive:

$$z^2 - 13z + 36 = 0 .$$

Its roots are: $z_1 = 4$ and $z_2 = 9$. Now we solve the equations:
$x^2 = 4$ and $x^2 = 9$. They have the roots correspondingly:
$x_1 = 2$, $x_2 = -2$, $x_3 = 3$; $x_4 = -3$. These numbers are the roots of the original equation (check this, please !).

Any equation of the shape: $ax^4 + bx^2 + c = 0$ *is called a* ***biquadratic equation.*** It is reduced to quadratic equations by using the exchange: $x^2 = z$.

E x a m p l e . Solve the *biquadratic* equation: $3x^4 - 123x^2 + 1200 = 0$.

S o l u t i o n . Exchanging: $x^2 = z$, and solving the equation:
$3z^2 - 123z + 1200 = 0$, we'll receive:

$$z = \frac{123 \pm \sqrt{123^2 - 4 \cdot 3 \cdot 1200}}{6} = \frac{123 \pm 27}{6};$$

hence, $z_1 = 25$ and $z_2 = 16$. Using our exchange, we receive:
$x^2 = 25$ and $x^2 = 16$, hence, $x_1 = 5$, $x_2 = -5$, $x_3 = 4$, $x_4 = -4$.

3. A *cubic equation* is the third degree equation; its general shape is:

$$ax^3 + bx^2 + cx + d = 0 .$$

The known Cardano's formulas for solution of this kind equations are very difficult and almost aren't used in practice. So, we recommend another way to solve the third degree equations.

a. At first we find one root of the equation by selecting, because these equations have always at least one real root, which is one of factors of a free term d; besides, coefficients of these equations have been selected usually so that the root, which must be found, is among not great integers, such as: -2, -1, 0, 1, 2 . Therefore, we'll find the root among these numbers and check it by substituting into the equation. A probability of successful result is very high. Assume this root is x_1 .

b. The second stage of solution is dividing of the third degree polynomial $ax^3 + bx^2 + cx + d$ by the binomial $(x - x_1)$. According to *Bezout's theorem* (see the section "Division of polynomial by linear binomial") this division is possible without a remainder, and we'll receive as a result the second degree polynomial, which would be annihilated, will give us a quadratic equation, solving which we'll find (or not !) the rest of the two roots.

E x a m p l e . Solve the equation: $x^3 - 3x^2 - 13x + 15 = 0$.

S o l u t i o n . Selecting the first root among the indicated numbers:

-2 , -1, 0, 1, 2 and substituting each of them in the given equation, we find that 1 is a root of this equation. Dividing the left-hand side of the equation by binomial $(x - 1)$, we'll receive:

$$
\begin{array}{l}
(x^3 - 3x^2 - 13x + 15) : (x - 1) = x^2 - 2x - 15 \\
-\ \ x^3 - x^2 \\
\overline{\quad -2x^2 - 13x} \\
\quad -\ \ -2x^2 + 2x \\
\quad \overline{\qquad -15x + 15} \\
\qquad -\ \ -15x + 15 \\
\qquad \overline{\qquad\quad 0}
\end{array}
$$

Now we solve a quadratic equation: $x^2 - 2x - 15 = 0$
find the rest of the two roots: $x_1 = -3$ and $x_2 = 5$.

Division of polynomial by linear binomial

Linear binomial is a polynomial of the first degree: $ax + b$. If to divide a polynomial, containing a letter x, by a linear binomial $x - b$, where b is a number (positive or negative), then a remainder will be a polynomial only of zero degree, i.e. some number N, which can be found without finding a quotient. Exactly, this number is equal to the value of the polynomial, received at $x = b$. This property is proved by *Bezout's theorem:* a polynomial $a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots + a_m$ *is divided by* $x - b$ *with a remainder* $N = a_0 b^m + a_1 b^{m-1} + a_2 b^{m-2} + \ldots + a_m$.

The p r o o f . According to the definition of division (see above) we have:

$$a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots + a_m = (x - b) Q + N ,$$

where Q is some polynomial, N is some number. Substitute here $x = b$, then $(x - b) Q$ will be missing and we receive:

$$a_0 b^m + a_0 b^{m-1} + a_0 b^{m-2} + \ldots + a_m = N .$$

The r e m a r k . It is possible, that $N = 0$. Then b is a root of the equation:

$$a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots + a_m = 0 .$$

Division of polynomials

Division of polynomials. What means to divide one polynomial P by another Q ? It means to find polynomials M (quotient) and N (remainder), satisfying the two requirements:

1). An equality $MQ + N = P$ takes place;
2). A degree of polynomial N is less than a degree of polynomial Q .

Division of polynomials can be done by the following scheme (*long division*):

$$\begin{array}{r} 4a + 3 \\ 4a^2 - a + 2 \overline{\smash{)}16a^3 + 8a^2 - 5a + 7} \end{array}$$

$$
\begin{array}{r}
-\underline{\ 16a^3 - 4a^2 + 8a\ } \\
12a^2 - 13a + 7 \\
-\underline{\ 12a^2 - 3a + 6\ } \\
10a\ \ + 1
\end{array}
$$

1) Divide the first term $16a^3$ of the dividend by the first term $4a^2$ of the divisor; the result $4a$ is the first term of the quotient.

2) Multiply the received term $4a$ by the divisor $4a^2 – a + 2$; write the result $16a^3 – 4a^2 + 8a$ under the dividend, one similar term under another.

3) Subtract terms of the result from the corresponding terms of the dividend and move down the next by the order term 7 of the dividend; the remainder is $12a^2 – 13a + 7$.

4) Divide the first term $12a^2$ of this expression by the first term $4a^2$ of the divisor; the result 3 is the second term of the quotient.

5) Multiply the received second term 3 by the divisor $4a^2 – a + 2$; write the result $12a^2 – 3a + 6$ again under the dividend, one similar term under another.

6) Subtract terms of the result from the corresponding terms of the previous remainder and receive the second remainder:
$– 10a + 1$. Its degree is less than the divisor degree; therefore the division has been finished. The quotient is $4a + 3$,
the remainder is $– 10a + 1$.

Linear equations in one unknown

An equation of the shape: $ax + b = 0$, where a and b – the known numbers, x – an unknown value, is called a ***linear equation in one unknown.*** To solve this equation means to find the numerical value of x, at which this equation becomes an identity.

If a is not equal to zero ($a \neq 0$), then a solution (root) has the shape:

$$x = -\frac{b}{a}.$$

If $a = 0$, then the two cases are possible:

1. $b = 0$, then $0 \cdot x + 0 = 0$. Here x can be *any number* (check this !).

2. $b \neq 0$, then $0 \cdot x + b = 0$. *There is no solution* (check this also).

E x a m p l e . Solve the equation: $\frac{x}{x-2} = \frac{x+1}{x+2}$. Multiply crosswise lied expressions: $x^2 + 2x = x^2 - 2x + x - 2$. Transfer all terms to the right-hand side of the equation. After reducing all similar terms we'll receive: $3x + 2 = 0$, hence $x = -2/3$.

Quadratic equation

A quadratic equation is an algebraic equation of the second degree:

$$ax^2 + bx + c = 0, \qquad (1)$$

where a, b, c – the given numerical or literal coefficients , x – an unknown.
If $a = 0$, then this equation becomes a linear one. Therefore, we'll consider here only $a \neq 0$. So, it is possible to divide all terms of the equation by a and then we receive:

$$x^2 + px + q = 0, \qquad (2)$$

where $p=b/a$, $q=c/a$. This quadratic equation is called a *reduced* one. The equation (1) is called a *non-reduced* quadratic equation. If b or c (or both) is equal to zero, then this equation is called a *pure* one. The examples of pure quadratic equations are following:

$$4x^2 - 12 = 0, \quad x^2 + 5x = 0, \quad x^2 = 36.$$

Main ways used at solving of equations

Solving of equation is a process, consisting mainly in a replacement of the given equation by another, equivalent equation. This replacement is called an ***identical transformation***. Main identical transformations are the following.

1. *Replacement of one expression by another, identically equal to it.* For example, the equation $(3x+2)^2 = 15x + 10$ may be replaced by the next equivalent equation: $9x^2 + 12x + 4 = 15x + 10$.
2. *Transferring terms of equation from one side to another with back signs.* So, in the previous equation we can transfer all terms from the right-hand side to the left with the sign "minus": $9x^2 + 12x + 4 - 15x - 10 = 0$, after this we receive: $9x^2 - 3x - 6 = 0$.
3. *Multiplication or division of both sides of equation by the same expression (number), not equal to zero.* This is very important, because *a new equation can be not equivalent to previous, if the expression, by which we multiply or divide, can be equal to zero.*

   E x a m p l e : The equation $x - 1 = 0$ has the single root $x = 1$.
   Multiplying it by $x - 3$, we receive the equation $(x - 1)(x - 3) = 0$, which has two roots: $x = 1$ and $x = 3$. The last value isn't a root for the given equation $x - 1 = 0$. This value is so called an ***extraneous root.*** And vice versa, division can result to a ***loss of roots***. In our case, if $(x - 1)(x - 3) = 0$ is the origin equation, then the root $x = 3$ will be lost at division of this equation by $x - 3$.

In the last equation (p.2) we can divide all terms by 3 (not zero!) and finally receive:

$$3x^2 - x - 2 = 0 .$$

This equation is equivalent to an original one:

$$(3x+2)^2 = 15x + 10$$

4. It is possible *to raise both sides of an equation to an odd power* and *to extract the odd degree root from both sides of an equation*.
   It is necessary to remember that:
   a) raising to an *even power* can result in *acquisition of extraneous roots;*
   b) a *wrong* extraction of *even degree root* can result in *loss of roots.*

   E x a m p l e s : The equation $7x = 35$ has the single root $x = 5$. Raising this equation to the second power, we receive the equation:

$$49x^2 = 1225,$$

having the two roots: $x = 5$ and $x = -5$. The last value is an extraneous root. A *wrong* extraction of square root from both sides of the equation $49x^2 = 1225$ results in $7x = 35$, and we lose the root: $x = -5$. A *right* extraction of this root leads to the equation: $|7x| = 35$, hence the two cases imply:

1) $7x = 35$, then $x = 5$; 2) $-7x = 35$, then $x = -5$.

Hence, at a *right* extraction of square root we don't lose roots of an equation. What means a *right* extraction of a root? Here we meet the notion of an *arithmetical root,* which is considered further in the section of the same name.

Solution of a quadratic equation

In general case of a non-reduced quadratic equation:

$$ax^2 + bx + c = 0,$$

its roots are found by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If to divide all terms of a non-reduced quadratic equation by a (is it possible ?), and to sign $b/a = p$ and $c/a = q$, then we'll receive the ***reduced*** quadratic equation:

$$x^2 + px + q = 0,$$

roots of which are calculated by the formula:

$$x = -p/2 \pm \sqrt{(p/2)^2 - q}.$$

E x a m p l e. $x^2 + 5x + 6 = 0$. Here $p = 5$, $q = 6$. Then we have:

$$x = -5/2 \pm \sqrt{(5/2)^2 - 6} = -5/2 \pm 1/2,$$

hence, $x_1 = -5/2 + 1/2 = -2$, $x_2 = -5/2 - 1/2 = -3$

Main ways of solving word problems

The process in solving verbal problems can be summarized in this way:

The 3 R's and the ESP of solving verbal problems

R EAD	the problem thoroughly
R EPRESENT	the unknown by means of a variable
R ELATE	the unknown to each other and the values given in the problem
E QUATE	form an equation using facts in the problem
S OLVE	the equation
P ROVE	the answers

READING: Reading mathematics is different from reading an ordinary story or newspaper. When we read a problem in mathematics, we must be sure we catch each word.

REPRESENTING THE UNKNOWNS: The unknown numbers in the problem can be represented in several ways. For example, if one number is thrice another, we could represent them as x and $3x$; we could also represent them as x and $\frac{1}{3}x$. Always choose the simplest representation.

RELATING THE UNKNOWNS: Look for the key words that translate into equals. Some of these words are: is, are was, make, and equals.

EQUATION FORMING: As the unknowns in the problem are correctly represented, then the meaning of the story can easily be obtained by expressing these into an equation.

SOLVING AND PROVING: We already know the methods in solving equationd and we can easily check the answers to see if they satisfy the problem.

I. Number Problems:

The number problems are the easiest to translate into equations since the relationships among the numbers are directly stated in the problems.

EXAMPLE 1: One number is two more than thrice another. Their sum is 30. Find the numbers.

Solution:
READ: Reading the problem thoroughly, we know two things about the numbers:

a) their sizes: one of them is two more than thrice the other; and
b) their sum: the sum is 30.

REPRESENT: If we represent the numbers using the first sentence, we have:

Let x = the first number
then: 3x + 2 = the other number

RELATE: The relationship between the numbers x and 3x + 2 and the other number, 30, gives us an equation.
EQUATE: x + 3x + 2 = 30 (Their sum is 30)
SOLVE: In solving the equation, we have

$$x + 3x + 2 = 30$$
$$4x + 2 = 30$$
$$4x = 28 \quad \Rightarrow \quad x = 7$$

therefore, the first number is 7 and the other number is 3x + 2 = 3(7) + 2 = 23
Answers: 7 and 23

PROVE: To prove that the numbers satisfy the problem, we have

a) their sum is 30: 7 + 23 = 30
b) 23 is two more than thrice 7: 3(7) + 2 = 21 + 2 = 23

The answers satisfy both the conditions of the problem.

EXAMPLE 2: The sum of two numbers is 29 and their difference is 5.

Solution:
READ: Reading the problem thoroughly, we know two things about the numbers:

a) their sum: the sum of the 2 numbers is 29 and
b) their difference: the difference of the 2 numbers is 5.

REPRESENT: Using the first sentence to represent the unknown numbers, we have:

Let x = one of the numbers.

RELATE: Then 29 – x = the other number
EQUATE: The second sentence gives us the equation.

$$x - (29 - x) = 5 \qquad \text{(Their difference is 5)}$$

SOLVE: Solving the equation, we have

$$x - (29 - x) = 5$$
$$x - 29 + x = 5$$
$$2x - 29 = 5$$
$$2x = 34 \quad \Rightarrow \quad x = 17$$

therefore the other number is

$$29 - x = 29 - 17 = 12$$

PROVE: We must show that the numbers satisfy both sentences.

a) Their sum is 29: 17 + 12 = 29
b) Their difference is 5: 17 – 12 = 5

II. Odd, Even, and Consecutive Integers

The word consecutive means following in order without interruption. As we know, an integer refers to a whole number. Hence, consecutive integers which follow in order without interruption.

If you know the sum of a certain number of consecutive integers or consecutive odd/even integers, then you have all the information you need to find the said integers.

EXAMPLE 1: The sum of three consecutive integers is 90. Find the integers.
SOLUTION:

Let x = the first integer
then $x + 1$ = the next consecutive integer
and $x + 2$ = the third consecutive integer
their sum is 90:

$$x + (x + 1) + (x + 2) = 90$$

manipulating the equation, we have

$$x + (x + 1) + (x + 2) = 90$$
$$x + x + 1 + x + 2 = 90$$
$$3x + 3 = 90$$
$$3x = 87$$
$$x = 29 \Rightarrow 1^{st} \text{ integer}$$
$$x + 1 = 30 \Rightarrow 2^{nd} \text{ consecutive integer}$$
$$x + 2 = 31 \Rightarrow 3^{rd} \text{ consecutive integer}$$

ANSWER: the consecutive integers are 29, 30, 31
PROOF: 29 + 30 + 31 = 90

EXAMPLE 2: Find three consecutive odd integers whose sum is 57.
SOLUTION:

Let x = the first odd integer
then $x + 2$ = the second odd integer
and $x + 4$ = the third odd integer
their sum is 57:

$$x + (x + 1) + (x + 2) = 90$$

manipulating the equation, we have

$$x + (x + 2) + (x + 4) = 57$$
$$x + x + 2 + x + 4 = 57$$
$$3x + 6 = 57$$
$$3x = 51$$
$$x = 17 \Rightarrow 1^{st} \text{ odd integer}$$

$x + 2 = 19 \Rightarrow$ 2nd odd integer
$x + 4 = 21 \Rightarrow$ 3rd odd integer

ANSWER: The consecutive odd integers are 17, 19, and 21.
PROOF: $17 + 19 + 21 = 57$

III. Digit Problems

In finding the value of a number, the position of each digit must be considered.
For example:

7 alone has a value of seven ones
7 in 75 has a value of seventy ones (seven tens)

Suppose you have an unknown two-digit number, instead of calling it simply *n* you can let x represent the tens digit and y the ones digit. Then the number may be represented as

$$10x + y$$

if we want to write the number with the <u>digits reversed</u>, then the new number formed would have y as the tens digit and x the ones digit. Hence, the number can be represented by

$$10y + x$$

EXAMPLE 1: The units digit in a two digit number is one more than twice the tens digit. Find the number if the sum of the digit is 7.
SOLUTION:

Let x = the tens digit
then $2x + 1$ = the units digit
and $10x + 2x + 1$ = the number

The sum of the digits is 7:

$x + 2x + 1 = 7$
$3x + 1 = 7$
$3x = 6$
$x = 2$ tens digit
$2x + 1 = 2(2) + 1 = 5$ units digit

The number is $10x + 2x + 1 = 10(2) = 2(2) + 1 = 20 + 4 + 1 = 25$
PROOF: 5 is one more than twice 2: $5 = 2(2) + 1$
The sum of the digits is 7 : $2 + 5 = 7$

EXAMPLE 2: The tens digit of a three-digit number is 0. The sum if the other two digits is 6. Interchanging the units and hundreds digits decreases the number by 396. Find the original number.
SOLUTION:

Let x = the units digit
then $6 - x$ = the hundreds digit
and $100(6 - x) + 10(0) + x$ = the original number

$100x + 10(0) + 6 - x$ = the reversed number

Interchanging the units and the hundreds digits decreases the number by 396:

$$100x + 10(0) + 6 - x = 100(6 - x) + 10(0) + x - 396$$
$$100x + 6 - x = 600 - 100x + x - 396$$
$$100x - x + 100x - x = 600 - 396 - 6$$
$$198x = 198$$
$$x = 1$$

The original number is $100(6 - x) + 10(0) + x = 100(6 - 1) + 1$
$= 100(5) + 1$
$= 500 + 1$
$= 501$

PROOF: The sum of the units and hundreds digits is 6: $1 + 5 = 6$
The reversed number which is 105 decreases the original number by 396:
$105 = 501 - 396$
$105 = 105$

IV. Age Problems

In dealing with age problems, it is important to keep in mind that the ages of different people change at the same rate. For example, after two years, all the people in the given problem are two years older than they were at first. Four years ago, all the people in the problem were four years younger. Also, it is easier if one makes a table showing the presentation for current ages in the problem, "future" ages (a number of years from now), and "past" ages (a number of years ago). If possible, represent the youngest present age by a single letter, than represent the ages. This process is illustrated in the following examples.

EXAMPLE 1: Alvin is now 20 years older than his son. In 10 years, he will be twice as old as his son's age. What are the present ages?

READ : Reading the problem thoroughly, we find a relationship between the present ages of Alvin and his son and the relationship of their ages 10 years from now.
REPRESENT: Using the relationship between the present ages, we have
Let x = the son's present age
RELATE: Then x + 20 = Alvin's present age
After 10 years, each age is increased by 10 : x + 10 and x + 30 as given in the table below.

	Now	Future (10 years from now)
Son	x	$x + 10$
Alvin	$x + 20$	$x + 30$

EQUATE: Using the second relationship: In 10 years, Alvin's age is twice his son's, we have

$x + 30 = 2\ (x{+}10)$

SOLVE: Manipulating the equation, we have

$x + 30 = 2\ (x + 10)$

$x + 30 = 2x + 20$

$x = 10$

ANSWERS: $x = 10$ Son's age

$x + 20 = 10 + 20 = 30$ Alvin's age

PROOF: Alvin's age is 20 years more than his son's: 30 = 10 + 20.

In 10 years:

Son: 10 + 10 = 20

Alvin: 30+10 = 40

Alvin's age is twice his son's:

40 = 2 (20)

40 = 40

EXAMPLE 2: The sum of Richard's age ang Ruel's age is 60. Nine years ago, Richard has twice as old as Ruel then. How old is Ruel?

READ : We find that we have a relationship between the present ages and their ages 9 years ago.

REPRESENT: Let x = Ruel's present age

RELATE: Then 60 – x = Richard's present age

We subtract 9 years from each as shown in the table below.

	Now	Past (9 years ago)
Ruel	x	$x – 9$
Richard	$60 - x$	$51 - x$

EQUATE: The second relationship gives our equation: Nine yars ago, Richard was twice as old as Ruel then.

$51 – x = 2\ (x – 9)$

SOLVE: Manipulating the equation, we have :

$51 – x = 2(x – 9)$

$51 – x = 2x – 18$

$3x = 69$

$x = 23$

ANSWERS: $x = 23$ Ruel's age

$60 – x = 37$ Richard's age

PROOF: The sum of their ages is 60: Ruel's age = 23

Richard's age = 37

Add 60

Nine years ago, Richard was twice as old as Ruel then:

$$37 - 9 = 2\,(23 - 9)$$
$$28 = 2\,(14)$$
$$28 = 28$$

V. Work Problems

Among the kinds of verbal problems that are solved by fractional equations are work problems. If we are going to consider a task as one job, then the rate of doing work can be represented as $\frac{1}{x}$, where x is the number of time units required to complete the job. That is, if a tailor sews a pair of pants in 3 hours, his rate is $\frac{1}{3}$ of the job per hour. If a faucet fills a tank in 35 minutes, its rate is $\frac{1}{35}$ of the job per minute. Another thing that we must know in doing work problems is that the amount of work done is the product of the rate and time,

Work = Rate x Time

Lastly, it is also important to know that the sum of the parts of a job done by different people or forces adds up to the whole job.

EXAMPLE 1: Richard can build a doghouse by himself in 3 days. Alvin can build the same doghouse in 6 days. How long would it take them if they worked together?

SOLUTION: We are looking for the number of days it would take Richard and Alvin if they worked together. We can represent the time for each by the same letter since they begin and end work at the same time.

	Rate	x Time	Work
Richard	$\frac{1}{3}$	x	$\frac{x}{3}$
Alvin	$\frac{1}{6}$	x	$\frac{x}{6}$

The work Richard and Alvin did together equals one whole job. $\Rightarrow \frac{x}{3} + \frac{x}{6} = 1$

Manipulating the equation, we have

$$2x + x = 6$$
$$3x = 6$$

ANSWER: x = 2 days
PROOF:

The work done by Richard is $\frac{1}{3}\cdot 2 = \frac{2}{3}$ of the job

The work done by Alvin is $\frac{1}{6}\cdot 2 = \frac{1}{3}$ of the job

Together, they have done $\frac{2}{3}+\frac{1}{3}=1$ whole job. Therefore, the answer satisfies the conditions of the problems.

EXAMPLE 2: The San Beda High School, awimming pool has two inlet pipes. One pipe can fill the pool in 6 hours, the other can fill it in 3 hours. The pool has one outlet pipe that can empty the pool in 4 hours. One day, when filling the pool after it was cleaned, the outlet pipe was left open by mistake. How long did it take to fill the pool?
SOLUTION: We now have three rates. Two are added (the inlet pipes) while the other pipe (the outlet pipe) is subtracted. We can represent the time for each by the same letter since they begin and work at the same time.

	Rate	x Time	Work
Inlet Pipe 1	$\frac{1}{6}$	x	$\frac{x}{6}$
Inlet Pipe 2	$\frac{1}{3}$	x	$\frac{x}{3}$
Outlet Pipe	$\frac{1}{4}$	x	$\frac{x}{4}$

$$\frac{x}{6}+\frac{x}{3}-\frac{x}{4}=1 \implies \text{(multiply both sides by 12)}\ 2x+4x-3x=12 \implies 3x=12$$

ANSWER: $x = 4$ hours
PROOF:

In 4 hours, inlet pipe 1 fills $\frac{1}{6}\cdot 4 = \frac{2}{3}$ of the pool

inlet pipe 2 fills $\frac{1}{3}\cdot 4 = \frac{4}{3}$ of the pool

In 4 hours, outlet pipe empties $\frac{1}{4} \cdot 4 = 1$ of the pool

So, $\frac{2}{3} + \frac{4}{3} - 1 = 1$ whole pool

VI. Distance Problems

When an object moves without changing its speed or rate, that object is said to be in uniform motion. The following examples below illustrate the types of problem involving uniform motion.

EXAMPLE 1: Motion in opposite direction

Mr. Honda and Mr. Toyota arrange to meet on the highway connecting their hometowns. Mr. Honda drives at 45 kph and Mr. Toyota at 35 kph. They leave their homes which are 120 kilometers apart at the same time. In how many hours will they meet?

SOLUTION: In this problem they both travel with the same length of time

Let t = the number of hours before the men meet

	Rate x	Time	Distance
Mr. Honda	45	t	$45t$
Mr. Toyota	35	t	$35t$

The distances they travel must add up to the 120 kilometers between the towns, that is

Mr. Honda's distance + Mr. Toyota's distance = Total distance

$45t + 35t = 120$

Manipulating the equation, we have

$$45t + 35t = 120$$

$$80t = 120 \Longrightarrow t = \frac{3}{2} \text{ or } 1\frac{1}{2} \text{ hours}$$

ANSWER: They meet after $1\frac{1}{2}$ hours driving.

PROOF:

After $1\frac{1}{2}$ hours, Mr. Honda has gone $\frac{3}{2}$ x 45 = 67.5 km.

Mr. Toyota has gone $\frac{3}{2}$ x 35 = 52.5 km

Total Distance 67.5 + 52.5 = 120 km

Thus, our answers satisfy the conditions of the problem.

EXAMPLE 2: Motion in the same directions

There are two trains. The Rabbit heads north on the expressway at 45 kph. Exactly 12 minutes after, the Panther follows at a steady speed of 54 kph. How long does it take the Panther to overtake the Rabbit?

SOLUTION: the Panther travels for 12 minutes less than (12 min = $\frac{1}{5}$ hr) because it travels 12 minutes later. It will be easier to represent the shorter time by a single literal number.

Let x = the Panther's time

then $x + \frac{1}{5}$ = the Rabbit's time

	Rate x	Time	Distance
Rabbit	45	$x + \frac{1}{5}$	$45(x + \frac{1}{5})$
Panther	54	x	$54x$

When the Panther overtakes the Rabbit, the distances will be equal.

$$45(x + \frac{1}{5}) = 54x$$

$$45x + 9 = 54x \implies 9x = 9 \implies x = 1 \text{ hour}$$

ANSWER: The Panther overtakes the Rabbit in 1 hour.
PROOF:

After 1 hour, the Rabbit has gone $45(1 + \frac{1}{5}) = 45 + 9 = 54$ km and the Panther has gone $54(1) = 54$ km

VII. Solution Problems

A solution is a homogenous mixture of two or more substances whose components are uniformly distributed all throughout. We usually think of one of the substances as being dissolved in the other. The substance being dissolved is the **solute**, the substance in which the solute is dissolved is called the solvent. For example when we add sugar to coffee, we say that sugar is dissolved in coffee. If we add iodine to alcohol, then the iodine is dissolved in alcohol. In the preceding examples, sugar and iodine are the solutes while coffee and alcohol are the solvents. Usually the solvent is greater in quantity than the solute.

Some solutions have special names. The most usual alloys are of metals that have been melted together. However if the solvent is alcohol , the mixture is called a **tincture**.

Solutions are usually labeled by the **percentages of the solutes**. This refers to the strength of the solution. That is, a solution that is 20% salt is twice as "strong" as a solution of 10% salt. If we add water to the solution to weaken it, the water that we added has 0% salt; if we add salt to the solution to strengthen or to increase its strength, the salt is 100% salt.

In solving solution problems, we label each solution by the percentage of solute in it, rather than by its value. When the amount of a given solution is multiplied by its strength, then we get the amount of solute it contains. For example, if we have 500grams of a 15% salt solution, then we have 500 x 0.15 = 75grams of salt. Hence to form an equation, we use the fact that the amount of the solute in each solution being added must be equal to the amount of solute in the combined solution. To summarize the steps in solving solution problems, we have

1) Label each solution with its corresponding percentage.
2) Give the total amount of the combined solution.
3) Form the equation by multiplying vertically.

EXAMPLE 1:
A chemist has 400g of salt solution that is 10% salt. How many grams of 20% salt solution must be added to obtain a 12% solution of salt?

SOLUTION:
The problem asks for the number of grams of 20% salt solution to be added.

Let x be the number of grams of 20% salt solution.

400 g	+	x g	=	400 + x g
10%		20%	=	12%

Multiplying vertically, we obtain

$$0.10(400) + 0.20(x) = 0.12(400 + x)$$

Multiplying the equation by 100 to eliminate the decimals, we have

$$10(400) + 20(x) = 12(400 + x)$$

Manipulating the equation, we have

$$4000 + 20x = 12(400 + x)$$
$$8x = 800$$
$$x = 100 \text{ grams}$$

ANSWER: x = 100 grams of 20% solution

PROOF: The amount of salt in the 10% solution is

$$0.10(400) = 400 \text{ grams}$$

The amount of salt in 100 grams of 20% solution is

$$0.20(100) = 20 \text{ grams}$$

Adding the two solutions, we have 60 grams of salt in a total solution of

$$400 + 100 = 500 \text{ grams}$$

$$\frac{60}{500} = \frac{12}{100} = 12\%$$

Thus, the answer satisfies the conditions of the problem.

EXAMPLE 2:
A chemist mixes a 200 L of a solution that is 60% acid with a 300 L of a solution that is 20% acid. What is the acid percentage of the mixture?

SOLUTION:

The problem asks the acid percentage of the mixture.

Let x = the acid percentage of the mixture.

200 L	+	300 L	=	500 L
60%		20%		x%

Note that the problem tells us that we have a 500 L mixture.

$$60(200) + 20(300) = x(500)$$
$$12{,}000 + 6{,}000 = 500x$$
$$18{,}000 = 500x$$
$$x = 36$$

Answer: x = 36%

VIII. Investment Problems

Problems which are concerned with amount of money invested at different rates of interest are called **investment problems.** Investment problems are percent problems since interest rates are expressed as percent. A percent is a fraction with a denominator of 100.

When we solve percent problems, we use the formula.

$$p = br$$

where *p* represents a part of whatever is being considered (percentage); *b* represents the base; that is, the whole of what is being considered; and *r* represents the rate which is expressed as percent.

EXAMPLE 1:
The Faculty Trust Fund is P200,000. Some of the money is invested at an annual rate of 4 percent and the rest is invested at an annual rate of 6 percent. If the income from both investments is both P9,600 a year, how much is invested at each rate?

Solution:

Let x = the amount invested at 4%
then 200,000 - x = the amount invested at 6%

	b ·	r =	p
Some of the money	x	.04	.04x
The rest of the money	200,000 - x	.06	.06(200,000) - x

Income from one investment + Income from the other investment = Total income from both investments

0.4x + .06(200,000 – x) = P9,600
4x + 1,200,000 – 6x = P960, 000
2x = P240, 000
x = P120, 000
200,000 - x = P80, 000

ANSWER:

Amount invested at 4%: P120,000
Amount invested at 6%: P80,000

PROOF:
4% of P120,000 = P4,800
6% of P80,000 = P4,800
Total Income = P9,600
Thus the answers satisfy the conditions of the problem.

EXAMPLE 2:
Anthony invested a certain amount of money at 5 percent per year, and an amount twice as large at 6 percent per year. The total annual income from the two investments was P4,250. Find the amount invested at each rate.

SOLUTION:
Let x = the amount invested at 5%
then 2x = the amount invested at 6%

	b ·	r =	p
Some of the money	x	.05	.05x
The rest of the money	200,000 - x	.06	.06(2x)

Income from one investment + Income from the other investment = Total income from both investments

.05x + .06(2x) = P4,250
.05x + .12x = P4,250
.17x = P4,250

x = P25, 000amount invested at 5%
2x = P50, 000amount invested at 6%