BASIC ARITHMETIC AND ALGEBRA POINTERS

Whole (natural) numbers

Natural numbers – numbers, which appear as a result of calculus of single subjects: peoples, animals, birds, trees, different wares and so on. Series of natural numbers: 1, 2, 3, 4, 5, … is continued endlessly and is called ***natural series***.

Arithmetical operations

Addition – an operation of finding a sum of some numbers: 11 + 6 = 17. Here 11 and 6 – *addends*, 17 – the *sum*. If addends are changed by places, a sum is saved the same: 11 + 6 = 17 and 6 + 11 = 17.

Subtraction – an operation of finding an addend by a sum and another addend: 17 – 6 = 11. Here 17 is a *minuend*, 6 – a *subtrahend*, 11 – the *difference*.

Multiplication. To multiply one number n (a multiplicand) by another m (a multiplier) means to repeat a multiplicand n as an addend m times. The result of multiplying is called a product. The operation of multiplication is written as: n x m or n · m. For example, 12 x 4 = 12 + 12 + 12 + 12 = 48. In our case 12 x 4 = 48 or 12 · 4 = 48. Here 12 is a multiplicand, 4 – a multiplier, 48 – a product. If a multiplicand n and a multiplier m are changed by places, their product is saved the same: 12 · 4 = 12 + 12 + 12 + 12 = = 48 and 4 ·12 = 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = 48. Therefore, a multiplicand and a multiplier are called usually *factors* or multipliers.

Division – an operation of finding one of factors by a product and another factor: 48 : 4 = 12. Here 48 is a *dividend*, 4 – a *divisor*, 12 – the *quotient*. At *dividing integers* a quotient can be not a whole number. Then this quotient can be present as a *fraction*. If a quotient is a whole number, then it is called that numbers are *divisible*, i.e. one number is divided *without remainder* by another. Otherwise, we have a division *with remainder*. For example, 23 isn't divided by 4 ; this case can be written as: 23 = 5 · 4 + 3. Here 3 is a *remainder*.

Raising to a power. To raise a number to a whole (second, third, forth, fifth etc.) *power* means to repeat it as a factor two, three, four, five and so on. The number, repeated as a factor, is called a *base of a power*; the quantity of factors is called an *index* or an *exponent of a power*; the result is called a *value of a power*. A raising to a power is written as:

$$3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243 .$$

Here 3 – a base of the power, 5 – an exponent (an index) of the power, 243 – a value of the power. The second power is called a *square*, the third one – a *cube*. The first power of any number is this number.

Extraction of a root – an operation of finding a base of a power by the power and its exponent:

$$\sqrt[5]{243} = 3 .$$

Here 243 – a *radicand*, 5 – an *index* (*degree*) *of the root*, 3 – a *value of the root*. The second root is called a *square root,* the third root – a *cube root*.The second degree of square root isn't written:

$$\sqrt{16} = 4 .$$

Addition and subtraction, multiplication and division, raising to a power and extraction of a root are two by two *mutually inverse operations*.

Order of operations. Brackets

If brackets are absent, the following *order of operations* is right:

1) raising to a power and extraction of a root (one after another);
2) multiplication and division (one after another);
3) addition and subtraction (one after another).

If brackets are present, at *first all operations inside brackets are executed* according to the aforesaid order, and then the rest of the operations out of brackets are executed (in the same order).

E x a m p l e. Calculate the next expression:

$$(10 + 2^3 \cdot 3) + 4^3 - (16 : 2 - 1) \cdot 5 - 150 : 5^2 .$$

S o l u t i o n. At first, powers must be calculated and changed by theirs values:

$$(10 + 8 \cdot 3) + 64 - (16: 2 - 1) \cdot 5 - 150: 25;$$

after this, multiplication and division in the brackets and out of them are executed:

$$(10 + 24) + 64 - (8 - 1) \cdot 5 - 6 ;$$

now, additions and subtractions in the brackets are executed:

$$34 + 64 - 7 \cdot 5 - 6 ;$$

finally, after the rest of the multiplication $7 \cdot 5 = 35$ we receive:

$$34 + 64 - 35 - 6 = 57$$

Laws of addition and multiplication

Commutative law of addition: *$m + n = n + m$. A sum isn't changed at rearrangement of its addends.*

Commutative law of multiplication: *$m \cdot n = n \cdot m$. A product isn't changed at rearrangement of its factors.*

Associative law of addition: *$(m + n) + k = m + (n + k) = m + n + k$. A sum doesn't depend on grouping of its addends.*

Associative law of multiplication: *$(m \cdot n) \cdot k = m \cdot (n \cdot k) = m \cdot n \cdot k$. A product doesn't depend on grouping of its factors.*

Distributive law of multiplication over addition: *$(m + n) \cdot k = m \cdot k + n \cdot k$. This law expands the rules of operations with brackets (see the previous section).*

Prime and composite numbers

Numbers, which aren't divisible by any numbers except 1 and itself, are called ***prime numbers.*** Numbers, which have also other factors, are called ***composite numbers.*** There is an infinite set of prime numbers. The set of them till 200 is:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43,

47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101,

103, 107, 109, 113, 127, 131, 137, 139, 149, 151,

157, 163, 167, 173, 179, 181, 191, 193, 197, 199.

Factorization. Resolution into prime factors

Any composite number can be presented as a product of prime factors by the single way. For example,

$$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3, \quad 225 = 3 \cdot 3 \cdot 5 \cdot 5, \quad 1050 = 2 \cdot 3 \cdot 5 \cdot 5 \cdot 7.$$

For small numbers this operation is easy. For large numbers it is possible to use the following way. Consider the number 1463. Look over prime numbers and stop, if the number is a factor of 1463. According to the divisibility criteria, we see that numbers 2, 3 and 5 aren't factors of 1463. But this number is divisible by 7, really, 1463: 7 = 209. By the same way we test the number 209 and find its factor: 209: 11 = 19. The last number is a prime one, so the found prime factors of 1463 are: 7, 11 and 19, i.e. $1463 = 7 \cdot 11 \cdot 19$. It is possible to write this process using the following record:

Number	Factor
1463	7
209	11
19	19

Greatest common factor

Common factor of some numbers - a number, which is a factor of each of them. For example, numbers 36, 60, 42 have common factors 2 and 3 . Among all common factors there is always the greatest one, in our case this is 6. This number is called a ***greatest common factor*** (GCF).

To find a ***greatest common factor*** (GCF) of some numbers it is necessary:

1) to express each of the numbers as a product of its *prime factors*, for example:

$$360 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 ,$$

2) to write *powers of all prime factors* in the factorization as:

$$360 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 2^3 \cdot 3^2 \cdot 5^1 ,$$

3) to write out all *common factors* in these factorizations;
4) to take *the least power* of each of them, meeting in the all factorizations;
5) to multiply these powers.

E x a m p l e . Find GCF for numbers: 168, 180 and 3024.

S o l u t i o n . $168 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 2^3 \cdot 3^1 \cdot 7^1$,

$$180 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 2^2 \cdot 3^2 \cdot 5^1,$$

$$3024 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 7 = 2^4 \cdot 3^3 \cdot 7^1.$$

Write out the least powers of the common factors 2 and 3 and multiply them:

$$GCF = 2^2 \cdot 3^1 = 12.$$

Least common multiple

Common multiple of some numbers is called a number, which is divisible by each of them. For example, numbers 9, 18 and 45 have as a common multiple 180. But 90 and 360 are also theirs common multiples. Among all common multiples there is always the least one, in our case this is 90. This number is called a ***least common multiple*** (LCM).

To find a ***least common multiple*** (LCM) of some numbers it is necessary:

1) to express each of the numbers as a product of its *prime factors*, for example:

$$504 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7,$$

2) to write *powers of all prime factors* in the factorization as:

$$504 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7 = 2^3 \cdot 3^2 \cdot 7^1,$$

3) to write out *all prime factors*, presented at least in one of these numbers;
4) to take *the greatest power* of each of them, meeting in the factorizations;
5) to multiply these powers.

E x a m p l e . Find LCM for numbers: 168, 180 and 3024.

S o l u t i o n . $168 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 2^3 \cdot 3^1 \cdot 7^1$,

$180 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 2^2 \cdot 3^2 \cdot 5^1$,

$3024 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 7 = 2^4 \cdot 3^3 \cdot 7^1$.

Write out the greatest powers of all prime factors: 2^4, 3^3, 5^1, 7^1
and multiply them:

$$\text{LCM} = 2^4 \cdot 3^3 \cdot 5 \cdot 7 = 15120 .$$

Divisibility criteria

Divisibility by 2. A number is divisible by 2, if its ***last digit*** is 0 or is divisible by 2. Numbers, which are divisible by 2 are called *even* numbers. Otherwise, numbers are called *odd* numbers.

Divisibility by 4. A number is divisible by 4, if its ***two last digits*** are zeros or they make a two-digit number, which is divisible by 4.

Divisibility by 8. A number is divisible by 8, if its ***three last digits*** are zeros or they make a three-digit number, which is divisible by 8.

Divisibility by 3 and by 9 . A number is divisible by 3, if ***a sum of its digits*** is divisible by 3. A number is divisible by 9, if ***a sum of its digits*** is divisible by 9.

Divisibility by 6. A number is divisible by 6, if it is divisible by 2 and by 3.

Divisibility by 5. A number is divisible by 5, if its ***last digit*** is 0 or 5.

Divisibility by 25. A number is divisible by 25, if its ***two last digits*** are zeros or they make a number, which is divisible by 25.

Divisibility by 10. A number is divisible by 10, if its ***last digit*** is 0.

Divisibility by 100. A number is divisible by 100, if its ***two last digits*** are zeros.

Divisibility by 1000. A number is divisible by 1000, if its ***three last digits*** are zeros.

Divisibility by 11. A number is divisible by 11 if and only if a sum of its digits, located on ***even places*** is equal to a sum of its digits, located on ***odd places***, OR these sums are differed by a number, which is divisible by 11.

There are criteria of divisibility for some other numbers, but these criteria are more difficult and not considered in a secondary school program.

E x a m p l e . A number 378015 is divisible by 3, because a sum of its digits 3 + 7 + 8 + 0 + 1 + 5 = 24, which is divisible by 3. This number is divisible by 5, because its last digit is 5. At last, this number is divisible by 11, because a sum of even digits: 7 + 0 + 5 =12 and a sum of odd digits: 3 + 8 + 1 = 12 are equal. But this number isn't divisible by 2, 4, 6, 8, 9, 10, 25, 100 and 1000, because … Check these cases yourself !

Simple fractions

A part of a unit or some equal parts of a unit is called a ***vulgar (simple) fraction***. A number of equal parts into which a unit has been divided, is called a *denominator*; a number of these taken parts, is called a *numerator*. A fraction record:

$$\frac{3}{7} \text{ or } 3/7.$$

Here 3 – a numerator, 7 – a denominator.
If a numerator is less than a denominator, then the fraction is less than 1 and called a *proper* fraction. If a numerator is equal to a denominator, the fraction is equal to 1. If a numerator is greater than a denominator, the fraction is greater than 1. In both last cases the fraction is called an *improper* fraction. If a numerator is divisible by a denominator, then this fraction is equal to a quotient: 63 / 7 = 9. If a division is executed with a remainder, then this improper fraction can be presented as a *mixed number:*

$$\frac{65}{7} = 9\,\frac{2}{7}.$$

Here 9 – an incomplete quotient (an integer part of the mixed number), 2 – a remainder (a numerator of the fractional part), 7 – a denominator .
It is often necessary to solve a reverse problem – to convert a mixed number into a fraction. For this purpose, multiply an integer part of a mixed number by a denominator and add a numerator of a fractional part. It will be a numerator of a vulgar fraction, and its denominator is saved the same.

E x a m p l e . Convert $8\frac{5}{9}$ into a vulgar fraction.

S o l u t i o n . 1) $8 \cdot 9 = 72$;

2) $72 + 5 = 77$;

3) $\frac{77}{9}$.

So, $8\frac{5}{9} = \frac{77}{9}$.

Reciprocal fractions are two fractions whose product is 1. For example, 3 / 7 and 7 / 3 ; 15 / 1 and 1 / 15 and so on.

Operations with simple fractions

Extension of a fraction. *A fraction value isn't changed, if to multiply its numerator and denominator by the same non-zero number.* This transformation of a fraction is called an *extension of a fraction.* For instance:

$$\frac{5}{9} = \frac{5 \cdot 7}{9 \cdot 7} = \frac{35}{63}; \qquad \frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{8}{12}.$$

Cancellation of a fraction. *A fraction value isn't changed, if to divide its numerator and denominator by the same non-zero number.* This transformation of a fraction is called a *cancellation of a fraction* or *lowest term.* For instance:

$$\frac{18}{27} = \frac{2 \cdot \cancel{9}}{3 \cdot \cancel{9}} = \frac{2}{3}; \qquad \frac{21}{28} = \frac{3 \cdot \cancel{7}}{4 \cdot \cancel{7}} = \frac{3}{4}.$$

Comparison of fractions. From two fractions with the same numerators that one is more, a denominator of which is less:

$$\frac{3}{5} > \frac{3}{7}; \qquad \frac{2}{9} < \frac{2}{3}.$$

From two fractions with the same denominators that one is more, a numerator of which is more:

$$\frac{3}{5} > \frac{2}{5}; \qquad \frac{5}{9} < \frac{7}{9}.$$

To compare two fractions, which have different both numerators and denominators, it is necessary to extend them to reduce to the same denominators.

E x a m p l e . Compare the fractions:

$$\frac{2}{3} \text{ and } \frac{7}{10}.$$

S o l u t i o n . Multiply numerator and denominator of the first fraction - by denominator of the second fraction and numerator and denominator of the second fraction - by denominator of the first fraction:

$$\frac{2}{3} = \frac{20}{30}; \quad \frac{7}{10} = \frac{21}{30}, \text{ and we see, that } \frac{2}{3} < \frac{7}{10}, \text{ because } \frac{20}{30} < \frac{21}{30}.$$

The used transformation of fractions is called a *reducing of fractions to a common denominator.*

Addition and subtraction of fractions. If denominators of fractions are the same, then in order to add the fractions it is necessary to add their numerators; in order to subtract the fractions it is necessary to subtract their numerators (in the same order). The received sum or difference will be a numerator of the result; a denominator is saved the same. If denominators of fractions are different, before these operations it is necessary to reduce fractions to a common denominator. At addition of mixed numbers a sum of integer parts and a sum of fractional parts are found separately. At subtracting mixed numbers we recommend at first to reduce the mixed numbers to improper fractions, then to subtract these fractions and after this to convert the result into a mixed number again (in case of need).

E x a m p l e.

$$7\frac{1}{4} - 4\frac{2}{3} = \frac{29}{4} - \frac{14}{3} = \frac{87}{12} - \frac{56}{12} = \frac{31}{12} = 2\frac{7}{12}.$$

Multiplication of fractions. To multiply some number by a fraction means to multiply it by a numerator and to divide a product by a denominator. Hence, we have the general rule for multiplication of fractions*: to multiply one fraction by another it is necessary to multiply separately their numerators and denominators and to divide the first product by the second.*

E x a m p l e .

$$\frac{2}{7} \cdot \frac{5}{9} = \frac{2 \cdot 5}{7 \cdot 9} = \frac{10}{63}.$$

Division of fractions. *To divide some number by a fraction it is necessary to multiply this number by a reciprocal fraction.* This rule follows from the definition of division (see the section "Arithmetical operations").

E x a m p l e .

$$\frac{3}{5} : \frac{12}{25} = \frac{3}{5} \cdot \frac{25}{12} = \frac{3 \cdot 25}{5 \cdot 12} = \frac{5}{4}.$$

Decimal fractions (decimals)

Decimal fraction is a result of dividing of unit by ten, hundred, thousand parts etc. These fractions are very comfortable in calculations, because they are based on the same system, that calculus and record of integers are built. Due to this both record and rules of operations with decimal fractions are actually the same as for integers.

$$9.5047 = 9 + \frac{5}{10} + \frac{0}{100} + \frac{4}{1000} + \frac{7}{10000}.$$

At recording decimal fractions it isn't necessary to mark parts (as denominator); this is known by place, that the corresponding digit occupies. At first the integer part of a number is written; to the right of it the decimal point is put; the first digit after the point means a number of tenths (a number of tenth parts of unit), the second – a number of hundredths, the third – thousandths, and so on. Digits, located after decimal point, are called *decimal places*.

E x a m p l e .

One of advantages of decimals – they are easily reduced to the shape of vulgar fractions: a number after a decimal point (5047 in our case) is a numerator, and the *n*-th power of 10 (*n* – a quantity of decimal places, in our case $n = 4$) is a denominator:

$$9.5047 = 9\frac{5047}{10000}.$$

If a decimal doesn't contain an integer part, zero is put before a decimal point:

$$\frac{13}{100} = 0.13.$$

Properties of decimals.

1. *A decimal fraction isn't changed, if to add some zeros to the right of it:*

13.6 =13.6000.

2. *A decimal fraction isn't changed, if to reject zeros, located in the end:*

0.00123000 = 0.00123

Note: *it's prohibited to reject zeros, located not in the end of a decimal!*

3. *A decimal fraction will be increased by 10, 100, 1000 ,…times, if to transfer a decimal point to one, two, three, … places to the right:*

3.675 ---> 367.5 (it increases by 100 times).

4. *A decimal fraction will be decreased by 10, 100, 1000 ,…times, if to transfer a decimal point to one, two, three, … places to the left:*

1536.78 ---> 1.53678 (it decreases by 1000 times)

These properties permit quickly to multiply and to divide decimal fractions by 10, 100, 1000 and so on.

Repeating decimal is a decimal in which a digit or a group of digits repeats endlessly in a pattern. This group of repeating digits is called a period *of decimal* and is written in brackets. For instance,

E x a m p l e. If to divide 47 by 11, then the result is 4.27272727… = 4.(27)

Operations with decimal fractions

Addition and subtraction of decimals. These operations are executed as well as an addition and a subtraction of whole numbers. It is only necessary to write the corresponding decimal places one under another.

E x a m p l e .

```
   3.07
+ 11.354
   0.009
--------
  14.433
```

Multiplication of decimals. At first stage let's multiply the fractions as integers, not taking a decimal point into consideration. After this we use the following rule: *a number of decimal places in a product is equal to a sum of numbers of decimal places in all factors.* **Note**: *<u>before putting the decimal point</u> in the product it is prohibited to reject zeros in the end of it !*

E x a m p l e .

```
    3.079
 ×  0.0064
 ---------
    12316
   18474
 ---------
   197056
```

A sum of numbers of decimal places in factors is equal: 3 + 4 = 7. A sum of digits in the product is 6. Therefore, it is necessary to add one zero to the left: 0197056 and to put before this a decimal point: *0.0197056*.

Division of decimals.

<u>Division of decimal fraction by integer.</u>

If a *dividend is less than a divisor*, write zero in an integer part of a quotient and put after it a decimal point. Then, not taking the decimal point of dividend into consideration, join to its integer part the next digit of fractional part and compare again the received integer part of a dividend with a divisor. If a new number is again less than a divisor, put one more zero after a decimal point in a quotient and join to an integer part of a dividend the next digit of its fractional part. Thus, repeat this process till the received dividend would be not more than a divisor. After this one can fulfill the division as for integers. If a *dividend is more than a divisor or equal to it*, divide at first its integer part, write a result of the division in the quotient and put a decimal point. After this one can continue the division as for integers.

E x a m p l e . Divide 1.328 by 64.

S o l u t i o n:

```
     0.02075
 64)1.328
   - 128
      48
   -  00
      480
      448
       320
   -   320
         0
```

Division of decimal fraction by another one.

At first transfer decimal points in a dividend and a divisor by the number of decimal places of divisor, i.e. make the divisor an integer. Now divide as well as in the previous case.

E x a m p l e . Divide 0.04569 by 0.0006.

S o l u t i o n. Transfer the decimal points to 4 places to the right and divide 456.9 by 6:

$$0.0006\overline{)0.04569} \longrightarrow \begin{array}{r} 76.15 \\ 6\overline{)456.9} \end{array}$$

Converting a decimal to a simple fraction and back

To convert a decimal to a vulgar fraction it is necessary: a number after a decimal point to make as the numerator, and the n-th power of 10 (here n – a quantity of decimal places) - as the denominator. A non-zero integer part of a decimal is saved the same in a vulgar fraction; a zero integer part is omitted. For example:

$$7.55 = 7\frac{55}{100} = 7\frac{11}{20}; \qquad 0.55 = \frac{55}{100} = \frac{11}{20}.$$

To convert a vulgar fraction to a decimal it is necessary to divide a numerator by a denominator according to the division rules.

E x a m p l e . Convert 5 / 8 to a decimal fraction.

S o l u t i o n . Dividing 5 by 8, we'll receive 0.625. (Check it, please !)

In the most of cases this process can be continued infinitely. Then a simple fraction cannot be converted exactly to a decimal. But in practice this is never required. Dividing is broken if decimal places, that are of interest, have been already received.

E x a m p l e . Convert 1 / 3 to a decimal fraction.

S o l u t i o n . Dividing 1 by 3 will be infinite: 1:3 = 0.3333… . Check it, please.

Percents

Percent *is a hundredth part of unit.* A record 1% means 0.01. There are three main problems by percents:

Problem 1. *Find an indicated percent of a given number.*
The given number is multiplied by the indicated number of percents; then a product is divided by 100.

E x a m p l e . A deposit in a bank has an annual increase 6%. A sum of money in the beginning was equal to $10000. How many dollars will the sum be increased by in the end of the year?

S o l u t i o n : $10000 · 6 / 100 = $600.

Problem 2. *Find a number by another given number and its percent value of the unknown number.*
The given number is divided by its percent value; the result is multiplied by 100.

E x a m p l e . A salary by January was equal to $15000, that was equal 7.5% of an annual salary. What was the annual salary ?

S o l u t i o n: $15000 / 7.5 · 100 = $200000.

Problem 3. *Find the percent expression of one number by another.*
The first number is divided by the second, and a result is multiplied by 100.

E x a m p l e . On 2001 a plant have produced 40000 cars; and on 2002 - only 36000 cars. What percent does it constitute relatively to the output of 2001 ?

S o l u t i o n: 36000 / 40000 · 100 = 90%.

Ratio and proportion. Proportionality

Ratio *is a quotient of dividing one number by another.*
Proportion *– an equality of two ratios.* For instance:

$$12 : 20 = 3 : 5; \quad a : b = c : d .$$

Border terms of the proportion: 12 and 5 in the first proportion;
a and *d* in the second proportion.

Middle terms of the proportion: 20 and 3 in the first proportion;
b and *c* in the second proportion.

The main property of a proportion: *A product of border terms of a proportion is equal to a product of its middle terms.*

Two mutually dependent values are called ***proportional*** *ones, if a ratio of their values is saved as invariable.* This invariable ratio of proportional values is called a ***factor of a proportionality***.

E x a m p l e . A mass of any substance is proportional to its volume. For instance, 2 liters of mercury weigh 27.2 kg, 5 liters weigh 68 kg, 7 liters weigh 95.2 kg. A ratio of mercury mass to its volume (factor of a proportionality) will be equal to:

$$\frac{27.2}{2} = \frac{68}{5} = \frac{95.2}{7} = 13.6 \text{ – a mercury density .}$$

Thus, a factor of a proportionality in this example is density.

The Integers and Rational Numbers

To the natural numbers one adjoins their negatives and zero to form the integers. The ratios *a*/*b* of the integers, where *a* and *b* are integers and *b* /= 0, constitute the rational numbers; the integers are those rational numbers for which *b* = 1. The rational numbers may also be represented by repeating decimals; e.g., 1/2 = 0.5000 …, 2/3 = 0.6666 …, 2/7 = 0.285714285714 …

Negative integers appear, when the greater integer is subtracted from the smaller one, for instance:

$$10 - 15 = -5$$

The sign "minus" before 5 shows, that this number is negative.

Series of negative integers continue endlessly:

$$-1,\ \ -2, -3, -4, -5 \ldots$$

Fractional negative numbers appear, for example, when the greater number is subtracted from the smaller one:

$$\frac{3}{5} - \frac{12}{5} = -\frac{9}{5}$$

Also it is possible to say, that fractional negative numbers appear as a result division of a negative integer by a natural number:

$$-13 \div 7 = -\frac{13}{7}$$

Positive numbers in contrast to ***negative numbers*** (integers and fractional ones), are the numbers, considered in arithmetic (also integers and fractional ones).

Rational numbers – positive and negative numbers (integers and fractional ones) and zero. The more exact definition of rational numbers, adopted in mathematics, is the following:

*A number is called **rational**, if it may be presented as a simple, not a cancelled fraction of the shape:* ***m /n***, *where m **and n are** integers.*

Irrational numbers

Irrational numbers** in contrast to **rational numbers** (see above) aren't presented as a vulgar, not cancelled fraction of the shape:* ***m / n *, where* ***m*** *and* ***n*** *are integers. There are numbers of a new kind*, which are calculated with any accuracy, but can't be changed by a rational number.
They can appear as results of geometrical measurements, for example:

- a ratio of a square diagonal length to its side length is equal to $\sqrt{2}$
- a ratio of a circumference length to its diameter length is an irrational number π.

Examples of another irrational numbers:

$$\sqrt{3}\,,\quad \sqrt[3]{25}\,,\quad \sqrt[3]{5+\sqrt{7}}\,,\quad e \text{ and so on.}$$

The Real Numbers

The real numbers are those representable by an infinite decimal expansion, which may be repeating or nonrepeating; they are in a one-to-one correspondence with the points on a straight line and are sometimes referred to as the continuum. Real numbers that have a nonrepeating decimal expansion are called irrational, i.e., they cannot be represented by any ratio of integers. The Greeks knew of the existence of irrational numbers through geometry; e.g., $\sqrt{2}$ is the length of the diagonal of a unit square. The proof that $\sqrt{2}$ is unable to be represented by such a ratio was the first proof of the existence of irrational numbers, and it caused tremendous upheaval in the mathematical thinking of that time.

Imaginary and complex numbers

Consider the pure quadratic equation:

$$x^2 = a\,,$$

where a – a known value. Its solution may be presented as:

$$x = \sqrt{a}\ .$$

Here the three cases are possible:

1). If $a = 0$, then $x = 0$.

2). If a is a positive number, then its square root has two values: one positive and one negative; for example, the equation $x^2 = 25$ has the two roots: 5 and –5.
This is often written as the root with double sign before: $x = \pm\sqrt{25}$.

3). If a – a negative number, then the equation has no solution among known us positive and negative numbers, because the second power of any number is a *non-negative* number (think over this!). But, if we wish to receive solutions of the equation $x^2 = a$ also at negative values of a, we are obliged to introduce the new kind numbers – ***imaginary numbers.*** So, a number is ***imaginary***, if its second power is a negative number. According to this definition of imaginary numbers we can define an *imaginary unit* as:

$$\sqrt{-1} = i, \text{ so that } i^2 = -1\ .$$

Then, for the equation

$$x^2 = -25$$

we receive the two *imaginary* roots: $x = \sqrt{-25} = 5i$ and $x = -\sqrt{-25} = -5i$. Substituting both these roots into our equation we'll receive the identity. Check it, please!

In contrast to imaginary numbers all the rest numbers (positive and negative, integers and fractional, rational and irrational ones) are called ***real numbers.*** A sum of a real and an imaginary number is called a ***complex number***, and marked as:

$$a + bi\ ,$$

where *a, b* – real numbers, *i* – an imaginary unit.
In more details about complex numbers see the section "Complex numbers".

E x a m p l e s of complex numbers: $3 + 4i$, $7 - 13.6i$, $0 + 25i = 25i$, $2 + \pi i$.

Monomials and polynomials

Monomial is a product of two or some factors, each of them is either a number, or a letter, or a power of a letter. For example,

$$3a^2b^4, \quad bd^3, \quad -17abc$$

are monomials. A single number or a single letter may be also considered as a monomial. Any factor of a monomial may be called a *coefficient*. Often only a *numerical factor* is called a *coefficient*. Monomials are called *similar* or *like* ones, if they are identical or differed only by coefficients. Therefore, if two or some monomials have identical letters or their powers, they are also similar (like) ones. *Degree of monomial* is a sum of exponents of the powers of all its letters.

Addition of monomials. If among a sum of monomials there are similar ones, he sum can be reduced to the more simple form:

$$ax^3y^2 - 5b^3x^3y^2 + c^5x^3y^2 = (a - 5b^3 + c^5)x^3y^2.$$

This operation is called *reducing of like terms.* Operation, done here, is called also *taking out of brackets.*

Multiplication of monomials. A product of some monomials can be simplified, only if it has powers of the same letters or numerical coefficients. In this case exponents of the powers are added and numerical coefficients are multiplied.
E x a m p l e :

$$5\ ax^3z^8(-7a^3x^3y^2) = -35a^4x^6y^2z^8.$$

Division of monomials. A quotient of two monomials can be simplified, if a dividend and a divisor have some powers of the same letters or numerical coefficients. In this case an exponent of the power in a divisor is subtracted from an exponent of the power in a dividend; a numerical coefficient of a dividend is divided by a numerical coefficient of a divisor.
E x a m p l e :

$$35\,a^4x^3z^9 : 7\,ax^2z^6 = 5\,a^3xz^3\,.$$

Polynomial is an algebraic sum of monomials. *Degree of polynomial is* the most of degrees of monomials, forming this polynomial.

Multiplication of sums and polynomials: a product of the sum of two or some expressions by any expression is equal to the sum of the products of each of the addends by this expression:

$$(\,p+q+r\,)\,a = pa+qa+ra \quad \text{- opening of brackets.}$$

Instead of the letters *p, q, r, a* any expressions can be taken.
E x a m p l e :

$$(\,x+y+z\,)(\,a+b\,) = x(\,a+b\,)+y(\,a+b\,)+z(\,a+b\,) = \\ = xa+xb+ya+yb+za+zb\,.$$

A product of sums is equal to the sum of all possible products of each addend of one sum to each addend of the other sum.

Algebraic fractions

Algebraic fraction is an expression of a shape *A / B*, where *A* and *B* can be a number, a monomial, a polynomial. As in arithmetic, *A* is called a *numerator*, *B* – a *denominator*. Arithmetical fraction is a particular case of an algebraic one.

Canceling fractions

E x a m p l e :

$$\frac{3a^2 - 5ab + 2b^2}{3a^2 + ab - 2b^2} = \frac{\cancel{(3a-2b)}(a-b)}{\cancel{(3a-2b)}(a+b)} = \frac{a-b}{a+b}$$

Addition and subtraction of fractions

To add or to subtract two or some fractions it is necessary to make the same operations as in arithmetic.
E x a m p l e :

$$\frac{a}{c^2x} + \frac{b}{cx^2} = \frac{ax + bc}{c^2x^2} .$$

Multiplication and division of fractions

Multiplication and division of algebraic fractions doesn't differ from the same operations in arithmetic. Canceling a fraction can be done both before and after multiplication of numerators and denominators.
E x a m p l e :

$$\frac{2ab^2}{cxy^2} : \frac{4a^2b}{3cy^3} = \frac{2ab^2 \cdot 3cy^3}{cxy^2 \cdot 4a^2b} = \frac{3by}{2ax} .$$